GEOMETRY OF PURE STATES OF N SPIN-J STYSTEM

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arXiv: 0910.3075, OSID (2010)

ABSTRACT

We present the geometry of pure states of an ensemble of N spin-J systems using a generalisation of the Majorana representation. The approach is based on Schur-Weyl duality that allows for simple interpretation of the state transformation under the action of general linear and permutation groups.

MAJORANA REPRESENTATION

Majorana representation [Majorana(1932)] allows one to uniquely represent spin-J state as $2J$ points on the Bloch-Poincare sphere.

The geometry is based on the stereographic projection:

- Arbitrary state $|z\rangle = [\cos(\theta/2), \sin(\theta/2)\exp(i\phi)]$ is connected with Bloch vector $\mathbf{n} = [\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta]$.
- The state of a spin-$\frac{1}{2}$ can be parametrized with a single complex number $z = e^{-i\phi}\cot\theta/2$.

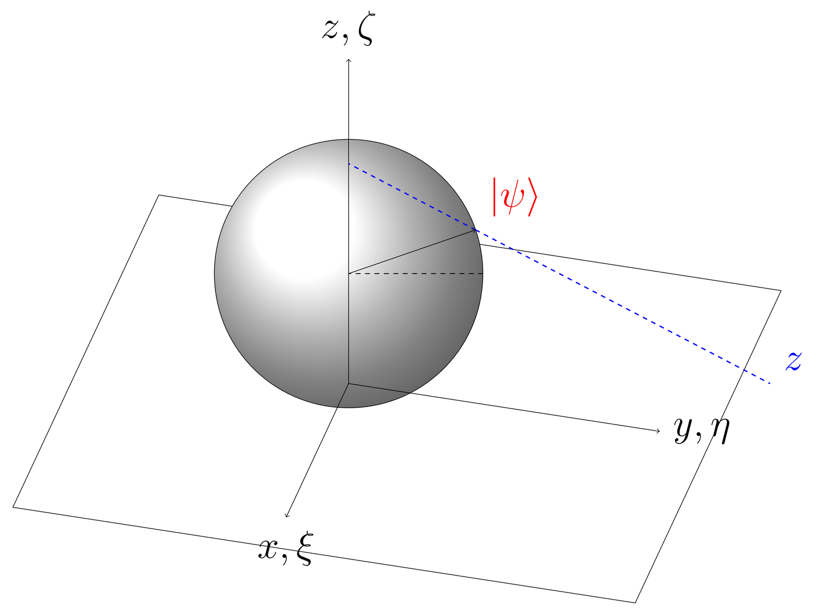

Figure 1: Stereographic projection

For a given state of spin-J:

$$|\psi\rangle = \sum_{m=-J}^{J} \psi_m |J, m\rangle$$

one can construct *Majorana polynomial*:

$$\mathrm{M}(|\psi\rangle; z) = \sum_{m=-J}^{J} (-1)^k \begin{pmatrix} 2J \\ J+m \end{pmatrix}^{\frac{1}{2}} \psi_m z^{J+m}.$$

For each spin-J state there exist a unique set of $2J$ complex numbers composed of $\tilde{N}$ roots of the Majorana polynomial $\{z_1, z_2, \ldots, z_{\tilde{N}}\}$ supplemented by $(2J - \tilde{N})$-element set of ∞. Resorting to stereographic projection each element of the set can be drawn on the Bloch-Poincare sphere. Examples are shown in figure 2.

(a) $|\psi\rangle = |6, -6\rangle + |6, -6\rangle$ **(b)** $|\psi\rangle = |3, -2\rangle + |3, 2\rangle$

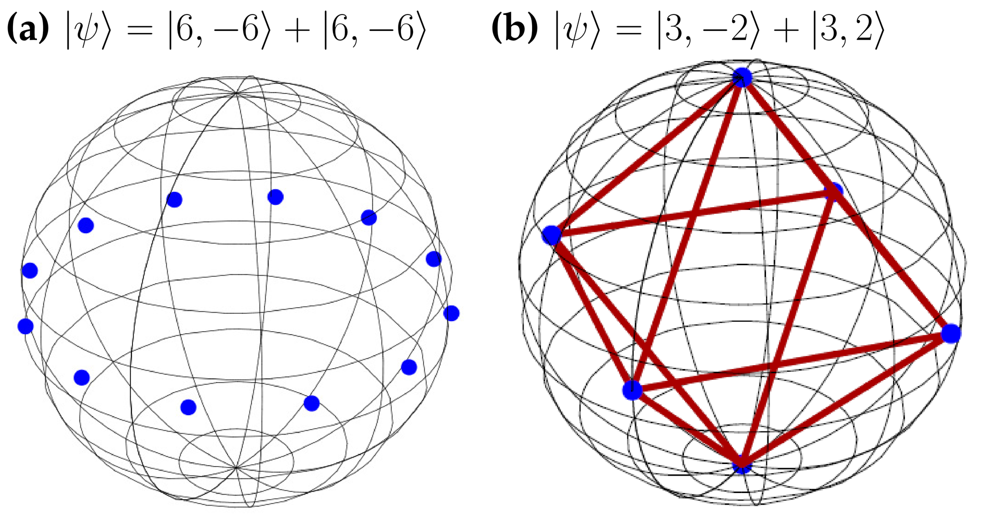

Figure 2: Majorana representation of (a) the NOON state and (b) the octahedron state (optimal for the local reference frames alignment [Kolenderski(2008)]).

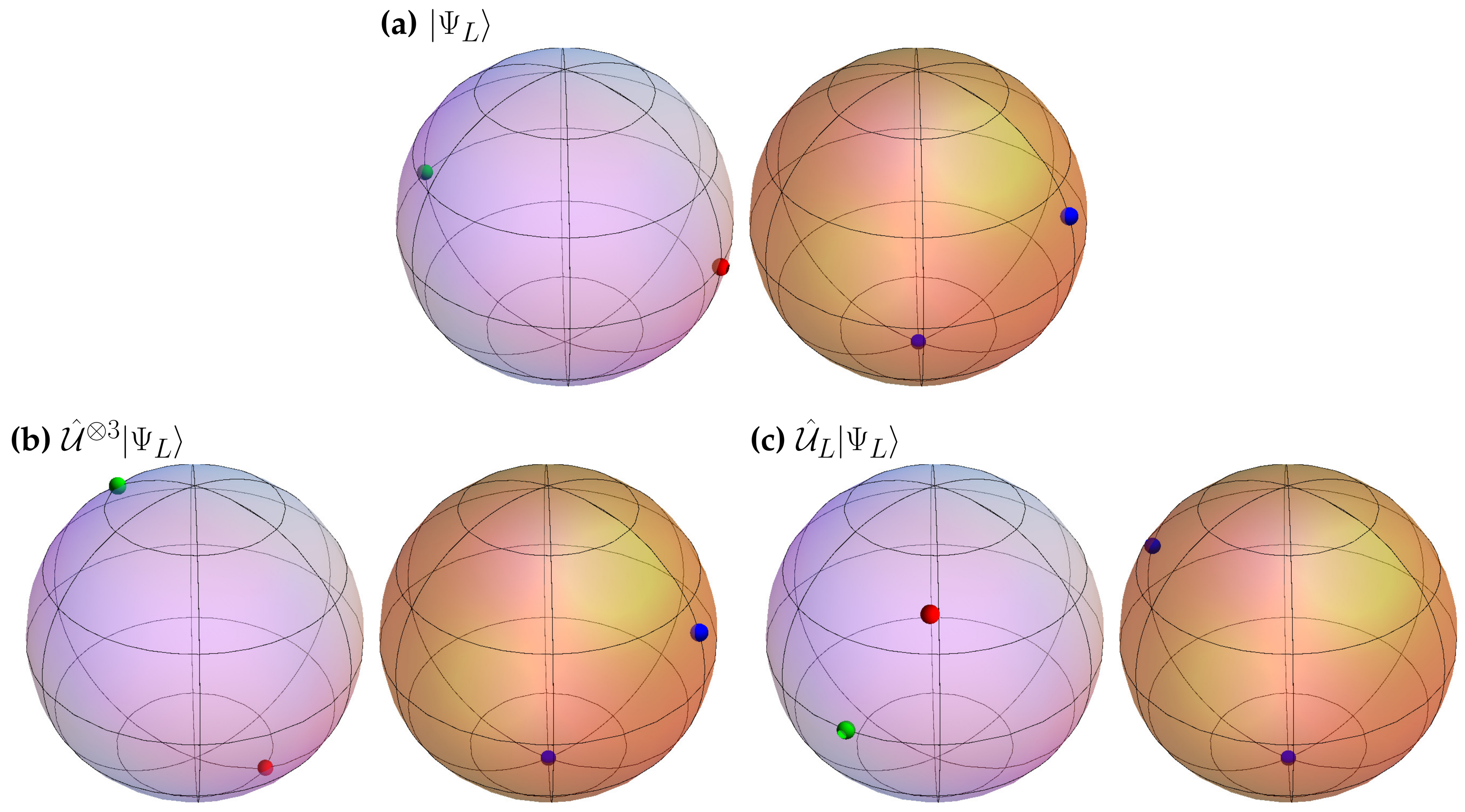

Figure 3: (a) The geometry of an exemplary state $|\Psi_L\rangle$ depicted in the representation (left) and multiplicity (right) spheres. (b) Under the action of $\hat{\mathcal{U}}^{\otimes 3}$ only the representation sphere experiences a modification. (c) The logical qubit transformation, in general, changes the representation states.

SCHUR-WEYL DUALITY

Let us consider a general linear group element $g \in GL(2J+1, \mathbb{C})$, permutation group element $s \in S_N$ and its respective representations $\hat{\mathcal{M}}(g)$, $\hat{\mathcal{S}}(s)$.

Theorem 1. *The joint action of general linear and permutation groups $\widehat{\mathcal{MS}}(g,s) = \hat{\mathcal{M}}(g)\hat{\mathcal{S}}(s)$ can be decomposed as:*

$$\widehat{\mathcal{MS}}(g,s) \cong \bigoplus_{\lambda \in Par(N,d)} \hat{\mathcal{M}}_\lambda(g) \otimes \hat{\mathcal{S}}_\lambda(s)$$

where $\hat{\mathcal{M}}_\lambda(g)$ and $\hat{\mathcal{S}}_\lambda(s)$ are irreducible representations (irreps) of $GL(2J+1, \mathbb{C})$ and S_N, respectively, and $Par(N,d)$ is a set of all partitions of N into d parts.

DECOMPOSITION

For simplicity, we consider here the case of N spin-$\frac{1}{2}$ (N qubits) and a unitary evolution ($SU(2)$). The Hilbert space of such system can be decomposed as:

$$\mathcal{H}_{J=\frac{1}{2}}^{\otimes N} = \bigoplus_{j=(N \bmod 2)/2}^{N/2} \mathcal{H}_j \otimes \mathbb{C}^{d_j}.$$

Moreover the action of unitary group is given by:

$$\hat{\mathcal{U}}(g)^{\otimes N}|\Psi\rangle = \sum_{j=(N \bmod 2)/2}^{N/2} \xi_j^\alpha \hat{\mathcal{U}}_j(g)|\psi_j^\alpha\rangle_j \otimes |\alpha\rangle_j,$$

where $\hat{\mathcal{U}}_j(g)$ is an irrep of $g \in SU(2)$. Accordingly, an arbitrary state of N qubits can be represented as:

$$|\Psi\rangle = \sum_{j=0(1/2)}^{N/2} \sum_{\alpha=0}^{d_j} \xi_{j\alpha}|\psi_{j\alpha}\rangle_j \otimes |\alpha\rangle_j.$$

It is seen that a state $|\Psi\rangle \in (\mathcal{H}_j)^{\otimes N}$ is in one to one correspondence with the :

- *representation states*: $|\psi_{j\alpha}\rangle_j \in \mathcal{H}_j$ and
- *multiplicity state*: $|\xi\rangle = \bigoplus_j \sum_\alpha \xi_j^\alpha |\alpha\rangle_j$.

Each of representation states and the multiplicity state can be represented graphically using Majorana representation. This allows us to depict the state of N qubits on:

- *representation sphere*, where all representation states $|\psi_{j\alpha}\rangle_j$ are drawn together and
- *multiplicity sphere*, drawing the multiplicity state $|\xi\rangle$.

EXEMPLARY APPLICATION

We consider here an exemplary application in the theory of decoherence free subspaces. We assume that the logical qubit is encoded into the state $|\Psi_L\rangle \in \mathcal{H}_{1/2}^{\otimes 3}$ of three physical qubits. In Majorana representation the action of the unitary rotation $\hat{\mathcal{U}}^{\otimes 3}$ can be seen as the rotation of all the points on the representation sphere as a solid body, whereas the points on the multiplicity sphere do not experience any modification. Hence *the logical qubit is entirely encoded in the multiplicity sphere and the „noisy evolution" is reflected in the representation sphere.*

For an exemplary state:

$$|\Psi_L\rangle = \frac{1}{2\sqrt{6}}\Big(2(|110\rangle + |001\rangle) - (1+\sqrt{3})(|101\rangle + |100\rangle) + (-1+\sqrt{3})(|011\rangle + |010\rangle)\Big)$$

one can easily find the multiplicity state:

$$|\xi\rangle = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

and representation states:

$$|\psi_{1/2}^0\rangle_{1/2} = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$
$$|\psi_{1/2}^1\rangle_{1/2} = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

A diagram summarizing the presented discussion is depicted in figure 3. The exemplary state $|\Psi_L\rangle$ and the action of the unitary $\hat{\mathcal{U}}^{\otimes 3}$ are depicted in panels (a) and (b), respectively. The last panel (c) shows the action of the unitary rotation of the logical qubit: $\hat{\mathcal{U}}_L(0, \xi_{1/2}^0, \xi_{1/2}^1) = (0, \xi_{1/2}^0 e^{i\alpha}, \xi_{1/2}^1 e^{-i\alpha})$, which in general affects the representation sphere.

The detailed discussion of the unitary and permutation group action can be found in Ref. [Kolenderski(2010)].

REFERENCES

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