QUANTUM GIROSCOPES AND PLATONIC SOLIDS

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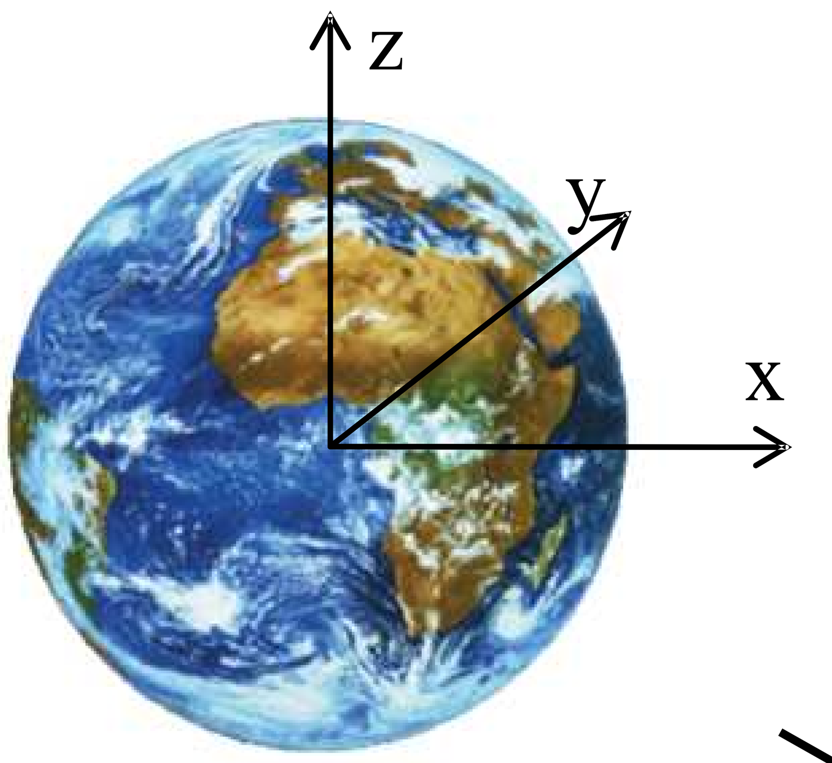
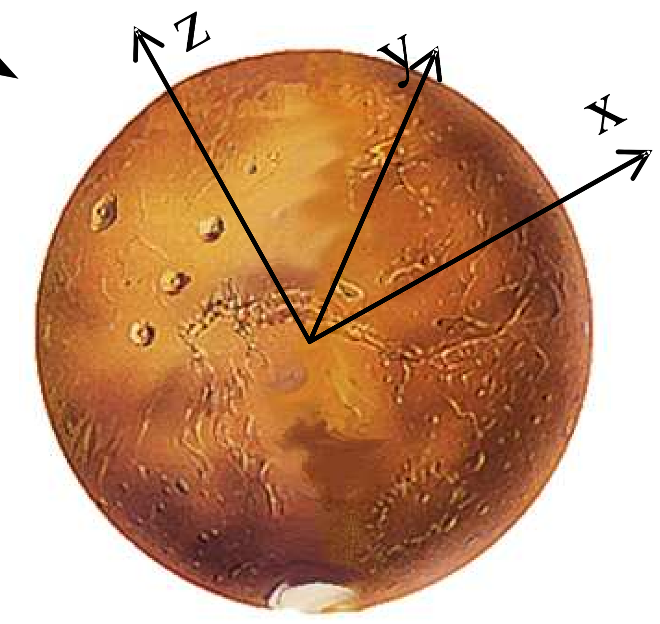
Problem

Consider two remote parties with slightly misaligned reference frames. Under given physically motivated restrictions, e.g. finite number of particles, limited energy, etc. find the state that yield the highest sensitivity to small misalignment of a cartesian reference frame.



$g \in \mathrm{SO}(3)$



Approach [Holevo(1982)]

One of the parties sends a quantum state $|\psi\rangle$ to the other party. After being sent the receiver has the state transformed by a unitary representation U_g , where g describes possible transformation of a cartesian reference frame. The transmitted state $|\psi_g\rangle = U_g|\psi\rangle$ is subsequently measured in order to get information regarding the transformation g.

The group element g is close to the known value $\tilde{g}$ and the goal is to find a measurement and the estimator featuring the highest sensitivity to small variations of g.

We parameterize the group element using Euler angles:

$$U_{g(\boldsymbol{\theta})} = e^{i\theta_x \boldsymbol{\sigma_x}} e^{i\theta_y \boldsymbol{\sigma_y}} e^{i\theta_z \boldsymbol{\sigma_z}}$$

We define a cost function $C(g, \tilde{g})$ penalizing for inaccurate estimation of g:

$$C(g(\tilde{\boldsymbol{\theta}}), g(\boldsymbol{\theta})) = (\Delta\theta_x)^2 + (\Delta\theta_y)^2 + (\Delta\theta_z)^2$$

Our optimization approach is based on the Cramér-Rao bound:

$$C(g(\tilde{\boldsymbol{\theta}}), g(\boldsymbol{\theta})) \geq \mathrm{Tr}(\boldsymbol{F}^{-1})/n,$$

where n is the number of repetitions of an experiment and $\boldsymbol{F}$ is the Fisher information matrix, which within our parametrization is simply the covariance matrix $\mathrm{Cov}_{\boldsymbol{J}}$ of spin $\boldsymbol{J}$:

$$\boldsymbol{F}_{ik} = 4\left(\frac{1}{2}\langle\psi|J_iJ_k + J_kJ_i|\psi\rangle - \langle\psi|J_i|\psi\rangle\langle\psi|J_k|\psi\rangle\right),$$

where $i, k = 1, 2, 3$ and $\boldsymbol{J} = \oplus_{j=0}^{N/2} \boldsymbol{J}^{(j)} \otimes \mathbb{1}$.

The optimal states are the ones minimizing the trace of the spin covariance matrix Tr($Cov_{\boldsymbol{J}}^{-1}$).

The problem above is hard to solve, however using relation between harmonic and arithmetic means $\mathrm{Tr}(\mathrm{Cov}_{\boldsymbol{J}}^{-1}) \geq 9/\mathrm{Tr}(\mathrm{Cov}_{\boldsymbol{J}})$ we may formulate a sufficient problem. The optimal state is the one:

- maximizing $\mathrm{Tr}(\mathrm{Cov}_{\boldsymbol{J}})$
- for which $\mathrm{Cov}_{\boldsymbol{J}} \propto \mathbb{1}$.

Both conditions are fulfilled simultaneously only for the states from fully symmetric subspace of $J = N/2$, which have equal projection deviations.

If the state is optimal, then:

$$\Delta^2 J_x = \Delta^2 J_y = \Delta^2 J_z = \frac{1}{3} N/2(N/2+1)$$

Examples of such states were introduced by [Zimba(2006)].

Majorana representation [Majorana(1932)]

A state from the fully symmetric subspace of N spins $1/2$:

$$|\psi\rangle = \sum_{m=-N/2}^{N/2} a_m |N/2, m\rangle$$

can always be written as a symmetrization of a product state of N spins $1/2$

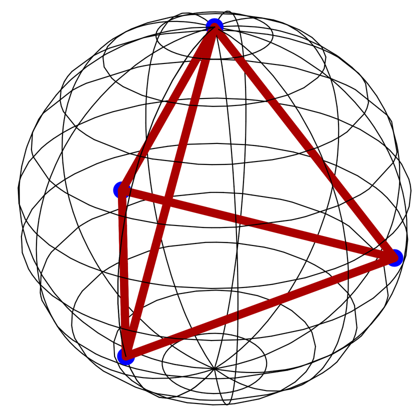
$$|\psi\rangle = \mathcal{N} \sum_{\sigma \in S_N} |\vec{n}_{\sigma(1)}\rangle \otimes |\vec{n}_{\sigma(2)}\rangle \otimes \cdots \otimes |\vec{n}_{\sigma(N)}\rangle.$$

The symmetric state of N spins $1/2$ is in one-to-one correspondence with N states of spin $1/2$ and as such may be represented as N points on Bloch sphere.

Solution

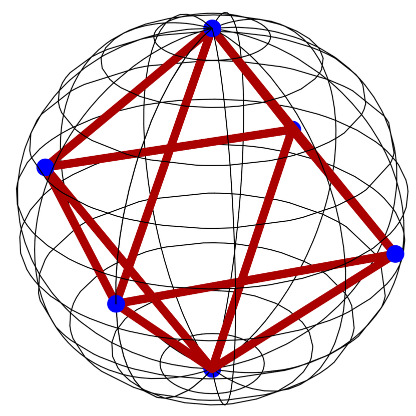
The states corresponding in Majorana representation to the platonic solids — tetrahedron, octahedron, cube, icosahedron and dodecahedron — are "the most sensitive" to arbitrary rotation.

(a) Tetrahedron type



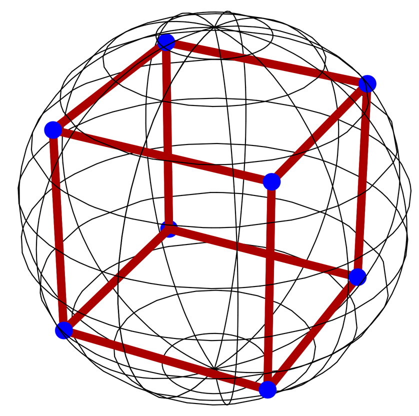
$|2, -2\rangle + \sqrt{2}|2, 1\rangle$

(b) Octahedron type



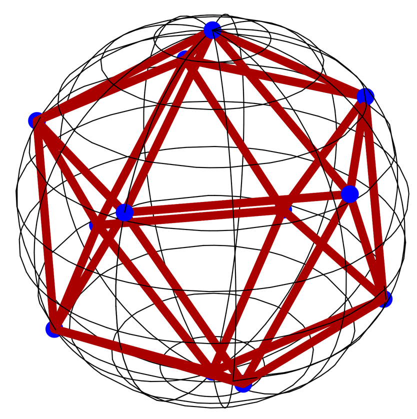
$|3, -2\rangle + |3, 2\rangle$

(c) Cube type



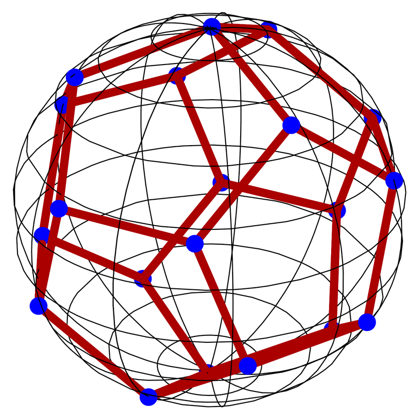
$|4, -4\rangle + \sqrt{\frac{14}{5}}|4, 0\rangle + |4, 4\rangle$

(d) Icosahedron type



$|6, -5\rangle + \sqrt{\frac{11}{7}}|6, 0\rangle - |6, 5\rangle$

(e) Dodecahedron type



$|10, -10\rangle + \sqrt{\frac{2}{5}}|10, -5\rangle + \sqrt{\frac{6}{5}}|10, 0\rangle - \sqrt{\frac{2}{5}}|10, 5\rangle - |10, 10\rangle$

Each of the above has its generalization for higher values of spin number.

References

[Holevo(1982)] A. S. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory* (North-Holland, Amsterdam, 1982).

[Zimba(2006)] J. Zimba, Electr. J. Theor. Phys **3**, 143 (2006).

[Majorana(1932)] E. Majorana, Nuovo Cimento **9**, 43 (1932).

Sponsors

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