QUANTUM TRAJECTORIES

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Schedule

- What is quantum trajectory?
- Stochastic unravelling
- Quantum state diffusion
- Example 1: Quantum chaos
- Example 2: Tunneling
- Benefits and questions

Quantum states

- quantum closed system S, pure state $|\psi(x_S, t)\rangle$

$$i\hbar \frac{\partial |\psi(x_S, t)\rangle}{\partial t} = H_S(x_S)|\psi(x_S, t)\rangle$$

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- reduced dynamics S+R $\longrightarrow$ S, an ensemble of time dependent pure states $\{|\psi(x_S,t)\rangle\}$

$$\rho(x_S,t) = \overline{|\psi(x_S,t)\rangle\langle\psi(x_S,t)|} \qquad \textit{density matrix}$$

unknown state of the environment $\rightarrow$ random properties of $|\psi(x_S,t)\rangle$

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a single member of the ensemble – a quantum trajectory

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- Who?
  - ⋆ D. F. Walls, C. W. Gardiner, P. Zoller
  - ⋆ G. J. Milburn
  - ⋆ H. J. Carmichael
  - ⋆ V. P. Belavkin
  - ⋆ N. Gisin, I. Percival

Quantum state diffusion

- quantum open system S+R – Schrödinger equation

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- reduced dynamics S+R $\longrightarrow$ S, von Neumann-Liouville equation for Born-Markov approximation (A. Kossakowski, G. Lindblad)

$$\frac{d}{dt}\rho \;=\; -\frac{i}{\hbar}[H_S,\rho] + \sum_k \left(L_k \rho L_k^\dagger - \frac{1}{2} L_k^\dagger L_k \rho - \frac{1}{2}\rho L_k^\dagger L_k \right)$$

operators L_k (Lindblads) depend on the way of the interaction with the environment

- unravelling – stochastic Schrödinger equation

$$|d\psi\rangle = |\psi(t+dt)\rangle - |\psi(t)\rangle = |drift\rangle\, dt + |fluctuations\rangle\, d\xi$$

$d\xi$ is a complex Wiener process with: $\overline{d\xi} = 0$, $\overline{Re(d\xi)Im(d\xi)}$, $\overline{Re(d\xi)Re(d\xi)} = \overline{Im(d\xi)Im(d\xi)} = dt$

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⋆ nonlinearity
⋆ nonlocality

Example 1: Quantum chaos in a kicked top

JI, P. Pepłowski, J.Phys. A **28**, 2183 (1995)

$$\dot{\rho} = -i[H_0 + H_1, \rho] + \Lambda\rho$$

$$H_0 = \beta J_z \qquad\qquad H_1 = \frac{\alpha}{j} \sum_k \delta(t-k)\, J_x^2$$

$$\Lambda\,\rho = \frac{\gamma_1}{2j}\Big([J_+, \rho J_-] + h.c.\Big) + \frac{\gamma_2}{2j}\Big([J_-, \rho J_+] + h.c.\Big)$$

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properties

$0 \leq \alpha \leq 1$ – stable north pole

$1 \leq \alpha \leq 2.25$ – two stable points tending to the equator

$2.25 \leq \alpha$ – strange attractor

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$|\psi\rangle$ - unnormalized

$|\psi\rangle = \sum_{n=-j}^{j} c_n\,|j,n\rangle$

$J_z\,|j,n\rangle = n\,|j,n\rangle$

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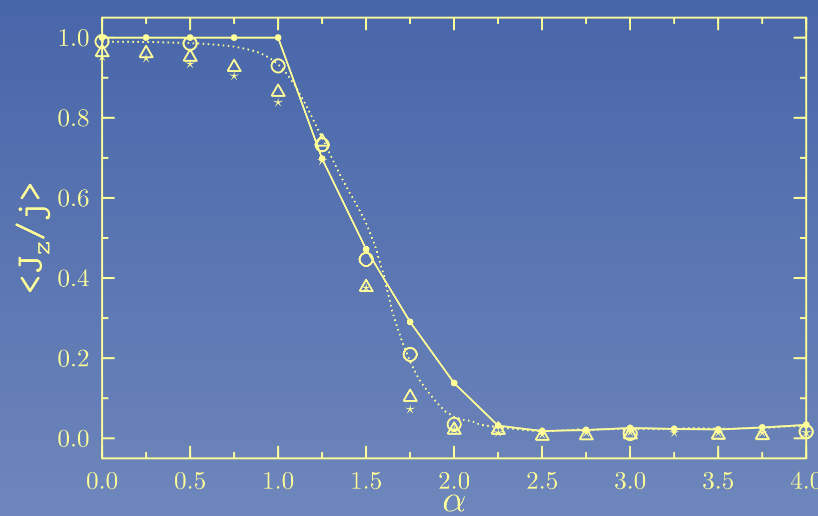
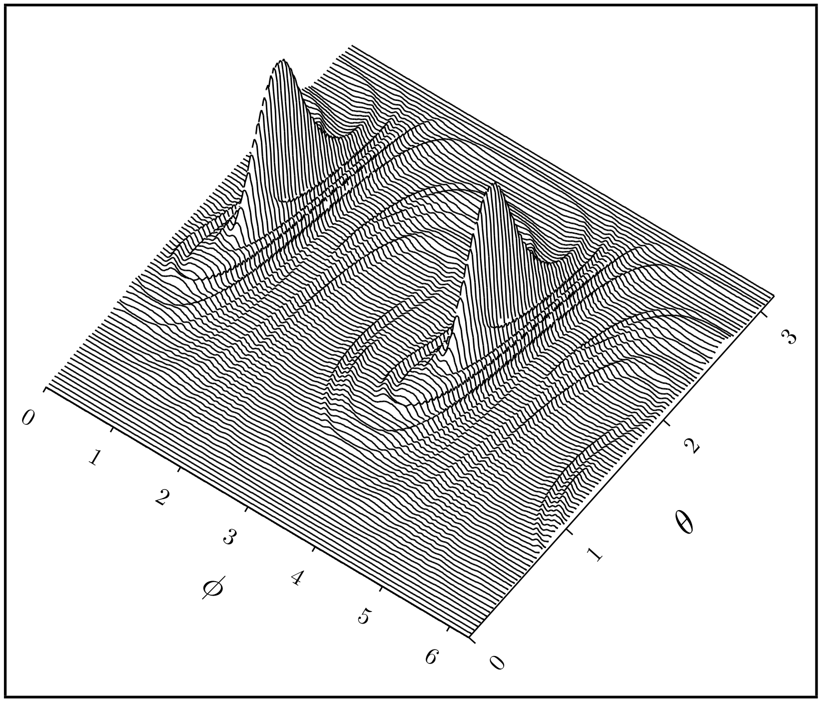


full line – classical, dotted – classical with noise, QSD – $j = 20$ (★), 30 (△), 100 (O)

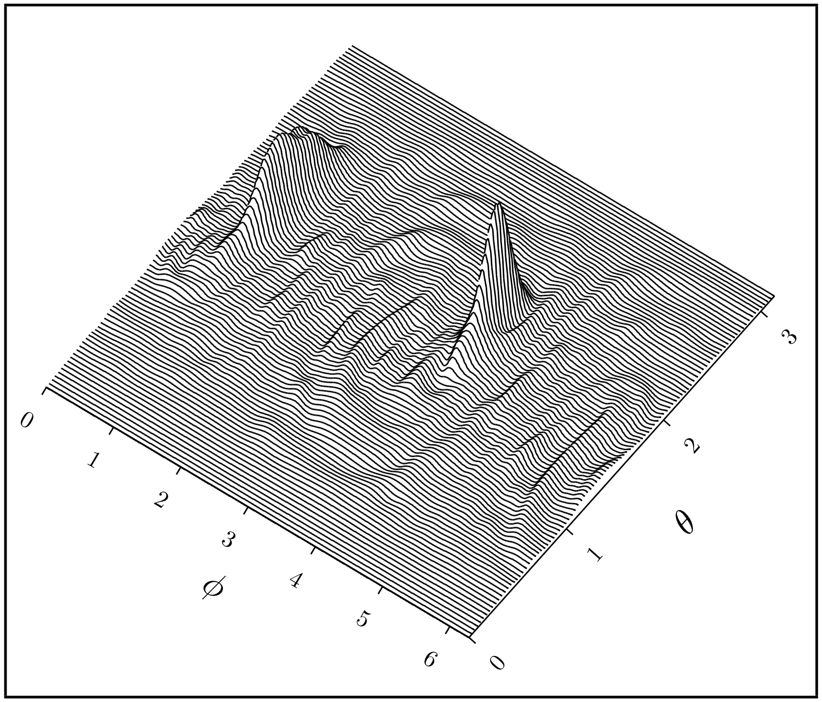
Regular motion $\alpha = 1.75$

classical



distribution function

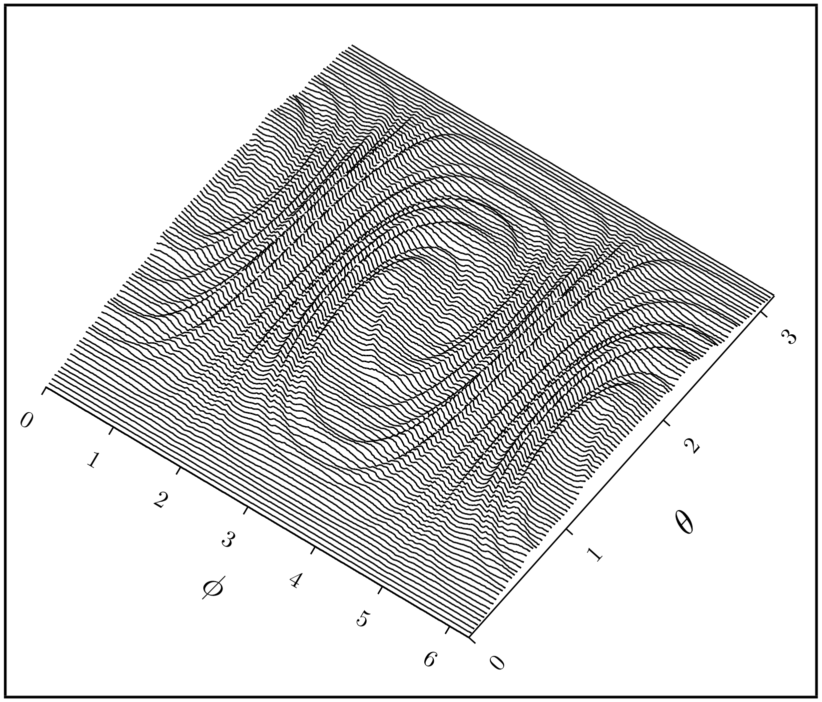
quantum



Q-representation ($j = 200$)

Chaotic motion $\alpha = 3.00$

classical



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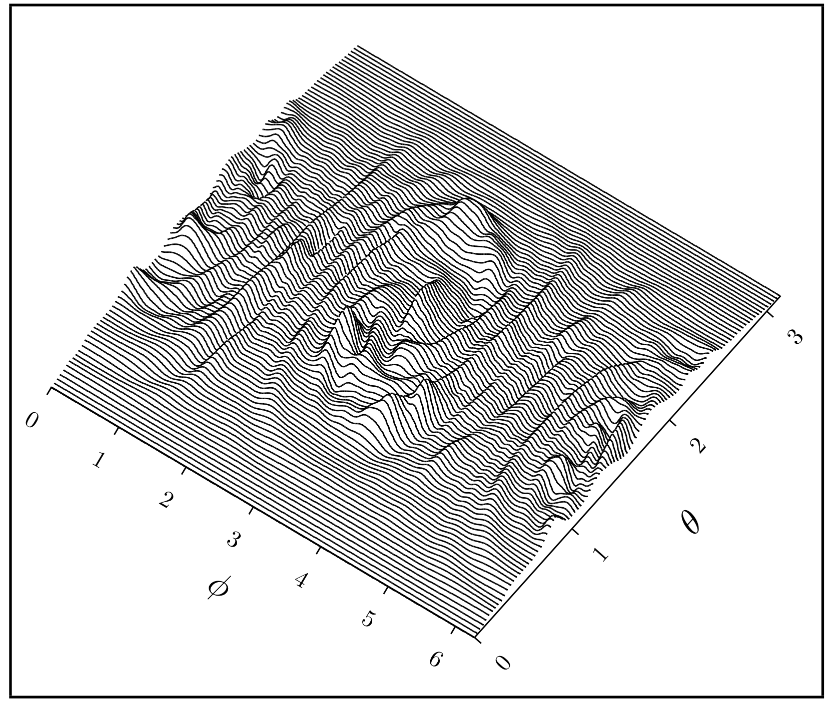

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Two-level approximation of a double-well potential

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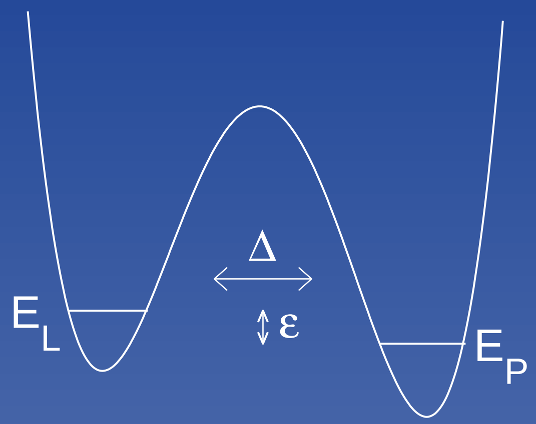

$$\epsilon = E_L - E_R$$

$$\Delta = \omega_0 \exp\left[-d\sqrt{2m\Delta U/\hbar}\right]$$

$$H_0(t) = -\frac{1}{2}\hbar\Delta\sigma_x + \frac{1}{2}\hbar\epsilon\sigma_z$$

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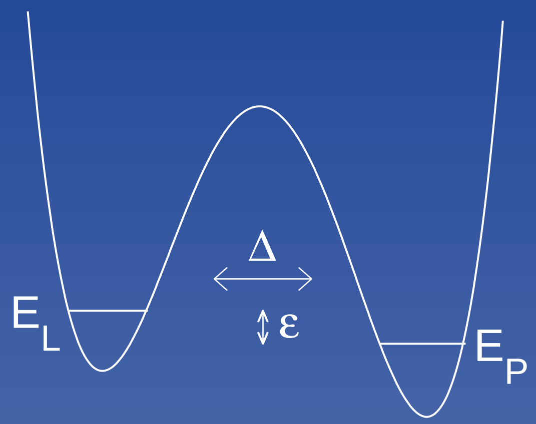

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Bloch vector

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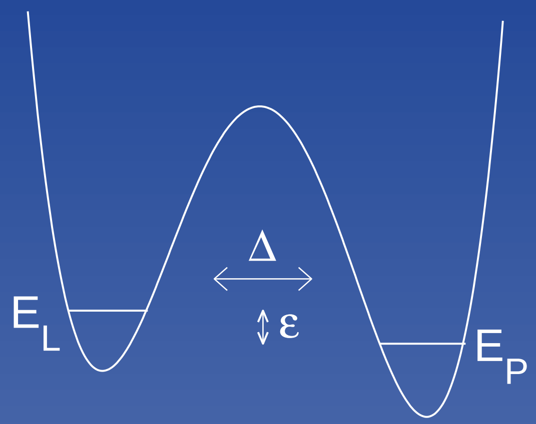

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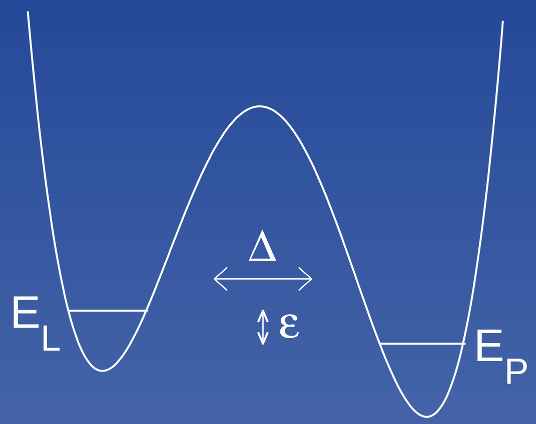

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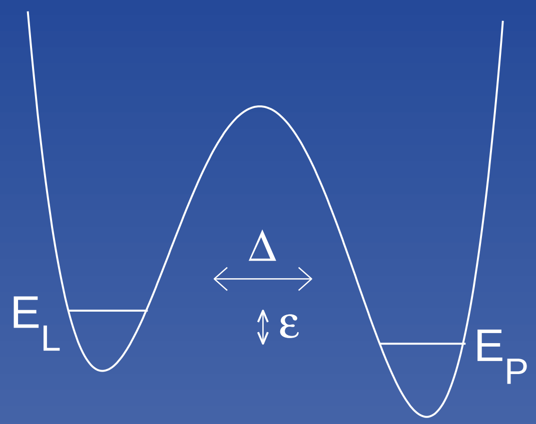

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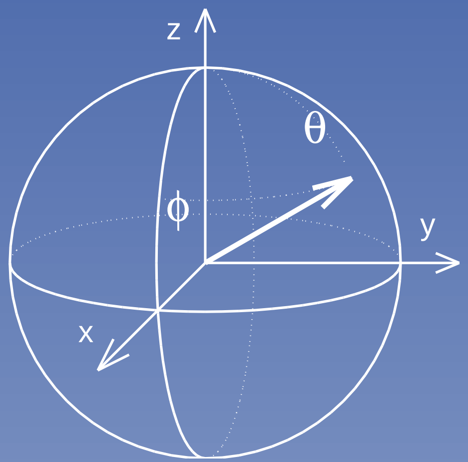

position $\sim \sigma_z \Rightarrow$ $$P_L(t) = \frac{1}{2}\left(\langle\sigma_z(t)\rangle + 1\right)$$

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Tunneling through a fluctuating barrier

JI, Phys.Rev. E **61**, 4890 (2000), Open. Sys. Information Dyn. **7**, 55 (2000)

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$\langle \eta(t) \rangle = 0\,, \quad \langle \eta(t)\eta(s) \rangle = \exp(-2\nu|t-s|)$

$$\nu = \frac{1}{2\tau}$$

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further on the biased case $\epsilon = 0$

von Neumann-Liouville evolution

$$\frac{d\vec{R}}{dt} = A\vec{R}, \quad \vec{R} = \begin{pmatrix} \langle \sigma_y(t) \rangle \\ \langle \sigma_z(t) \rangle \\ \langle \sigma_y(t)\eta(t) \rangle \\ \langle \sigma_z(t)\eta(t) \rangle \end{pmatrix}, \quad A = \begin{pmatrix} -2\kappa & \Delta_0 & 0 & \Delta_1 \\ -\Delta_0 & 0 & -\Delta_1 & 0 \\ 0 & \Delta_1 & -2\kappa - 2\nu & \Delta_0 \\ -\Delta_1 & 0 & -\Delta_0 & -2\nu \end{pmatrix}$$

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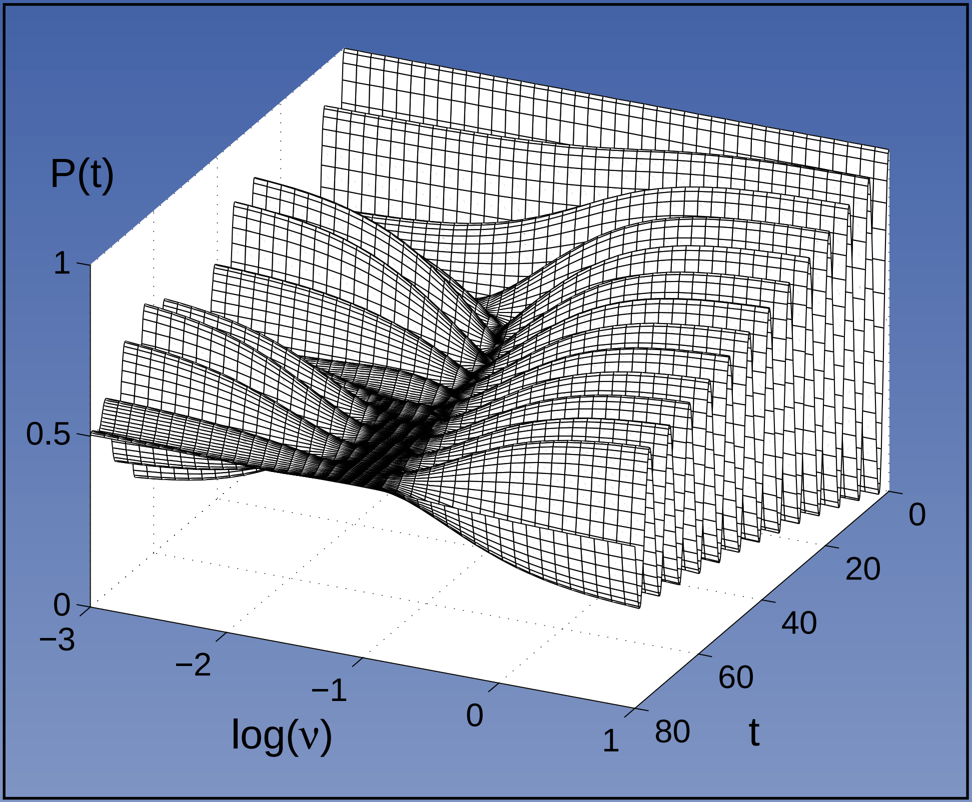

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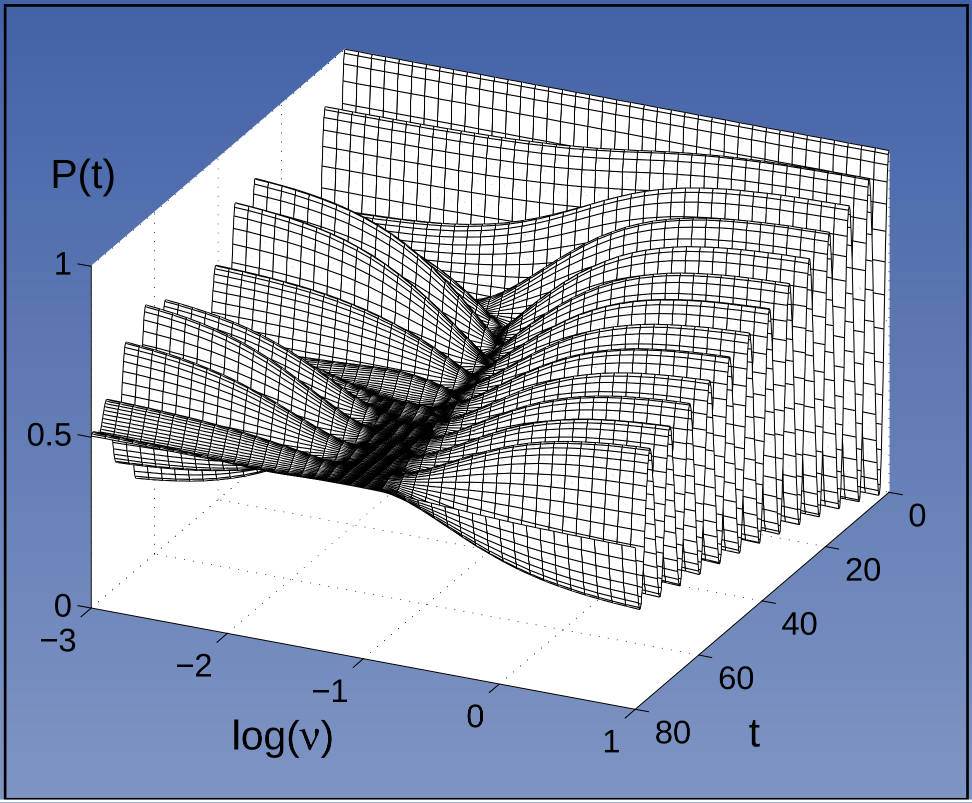

resonant suppression of coherence

$\nu = \Delta_1$

WHY?

stochastic evolution

$$d\phi = (\epsilon + \Delta(t)\cos(\phi)\cot(\theta))\,dt - \sqrt{\kappa} Im(d\xi)$$

$$d\theta = \left(\Delta(t)\sin(\phi) - \frac{\kappa}{2}\sin(\theta)\cos(\theta)\right)dt - \sqrt{\kappa}\sin(\theta) Re(d\xi)$$

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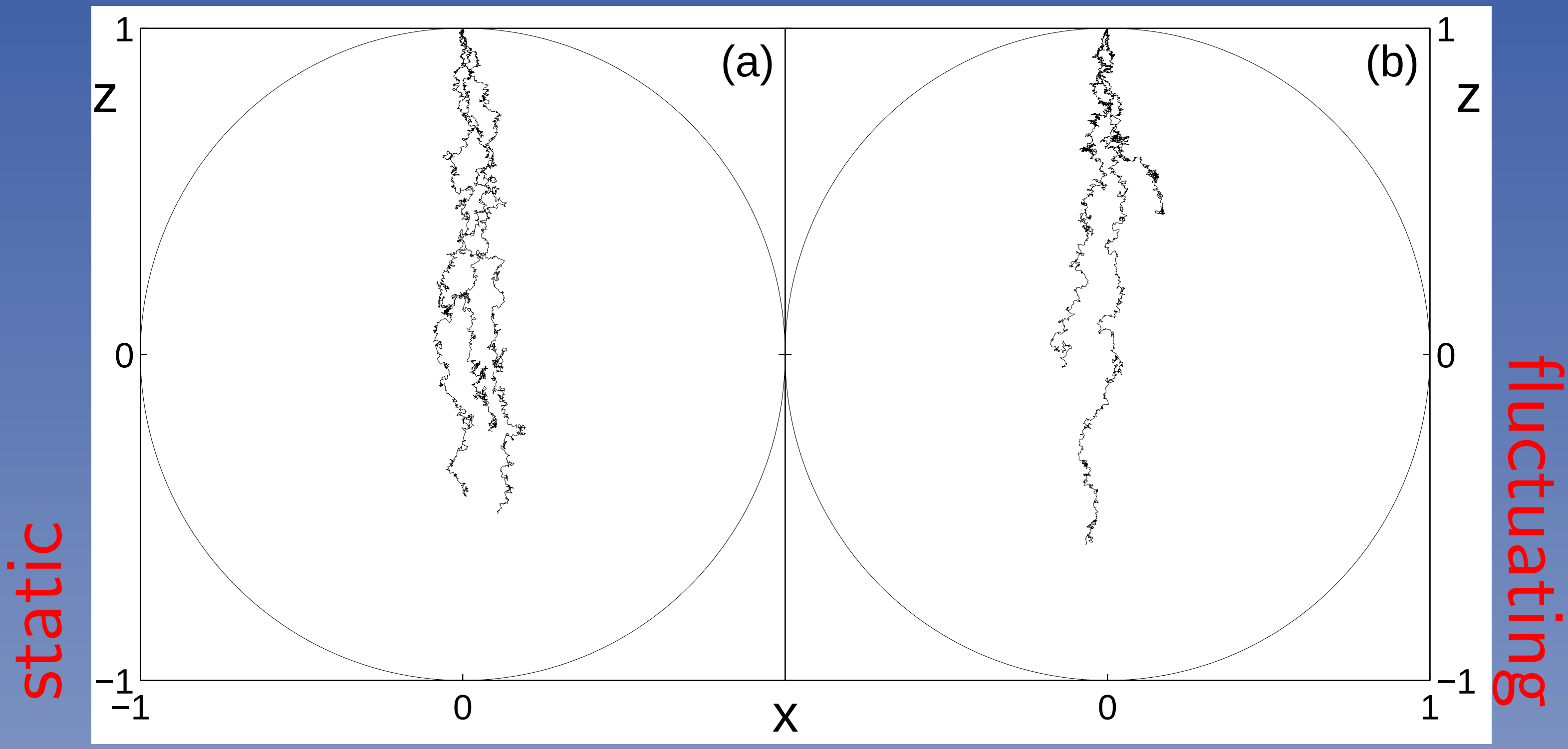


Mollweide projection

time static fluctuating

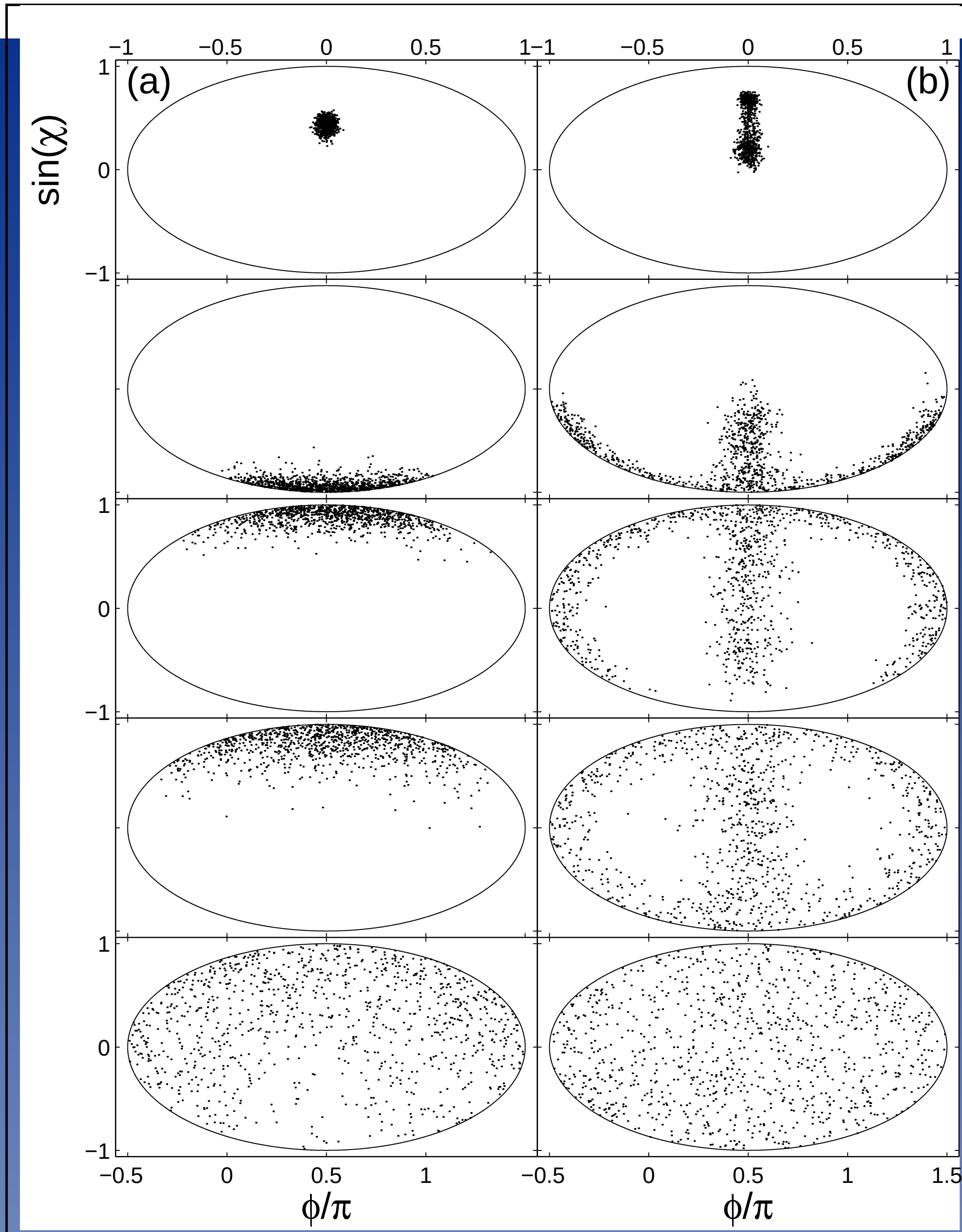
1

π

2π

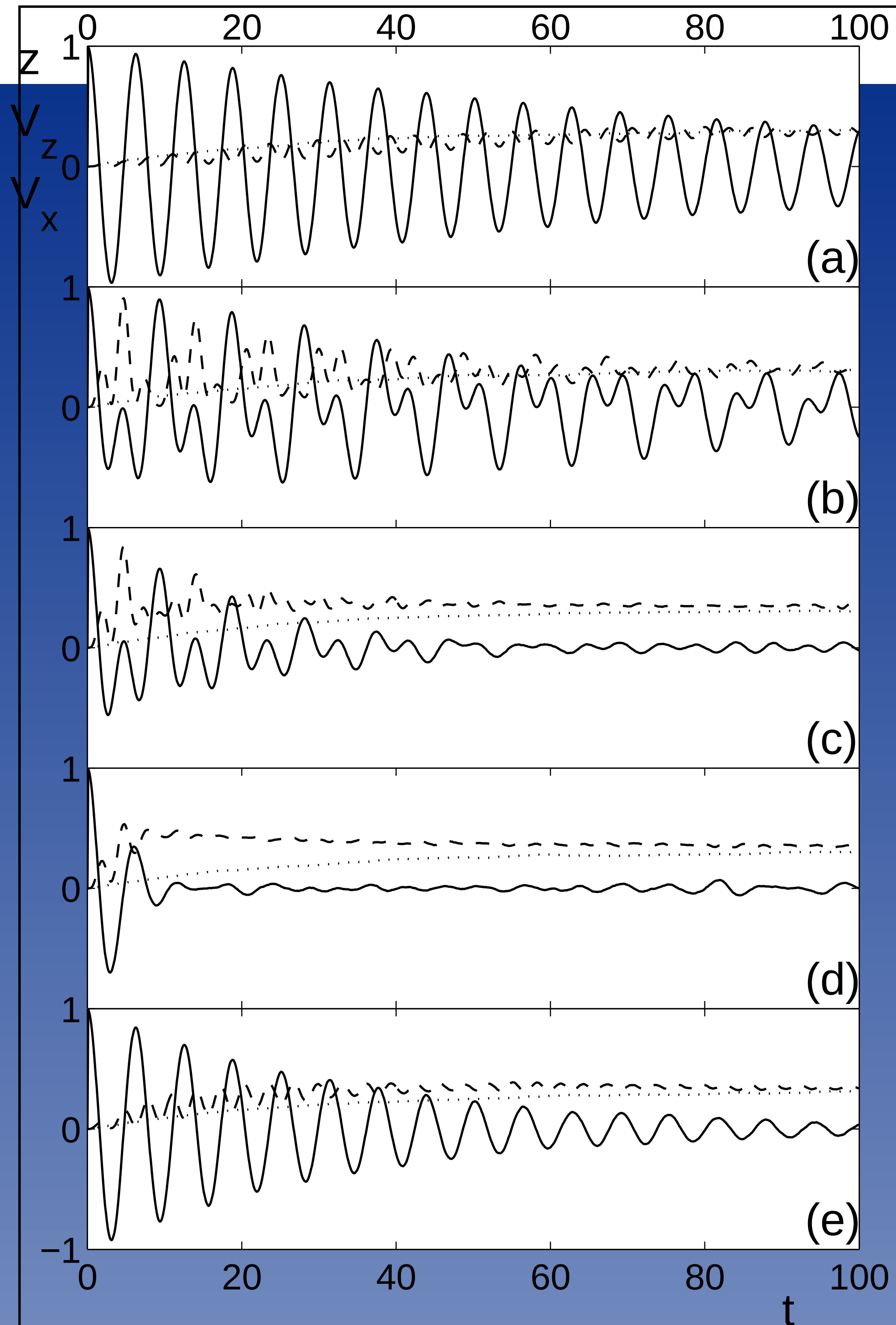
4π

100



Mean and variance

$\langle \sigma_z \rangle$ – continuous, $var(\sigma_z)$ – dashed, $var(\sigma_x)$ – dotted



$\nu = 0 \quad 0.035 \quad 0.35 \quad 3.5 \quad \infty$

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- ambiguity
- nonmarkovian bath
- strong coupling
- relativistic physics

Benefits and questions

Advantages:

- single realization - insight into the very foundations of quantum physics
  - ⋆ localization
  - ⋆ decoherence
  - ⋆ correspondence principle
- quantum jumps – absorption and emission
- measurement – the role of apparatus
- numerics

Problems

- ambiguity
- nonmarkovian bath
- strong coupling
- relativistic physics

THE END