THE VANDERMONDE DETERMINANT REVISITED

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Dedicated to the memory of Claude Itzykson (1938-1995)

The expansion of the even powers of the Vandermonde determinant in terms of signed sequences of Schur functions is considered. The $q-$discriminant is introduced to provide an explanation for the vanishing of certain expansion coefficients. A number of compelling conjectures are given. Finally connections with Hankel hyperdeterminants are noted.

In most sciences one generation tears down what another has built, and what one has established, another undoes. In mathematics alone each generation adds a new storey to the old structure. (Hermann Hankel 1839-1873)

1. Introduction

The Vandermonde determinant plays a crucial role in the description of the Quantum Hall Effect (QHE) via the Laughlin wavefunction ansatz[1] and in the description of One Component Plasmas[2] (OCP). There has been considerable interest in the expansion of the Laughlin wavefunction as a linear combination of Slater determinantal wavefunctions for N particles [3,4].

The *even* powers of the Vandermonde alternating function play a key role in determining the coefficients of the expansion of the Laughlin wavefunction as a linear combination of Slater determinantal wavefunctions. Indeed, the relevant coefficients are directly related to the signed integer coefficients that arise in the expansion of the even powers of the Vandermonde alternating function into Schur functions. The problem of determining the expansion coefficients for increasingly large values of N is a combinatorially explosive problem.

The primary problem is to determine the signed integer coefficients that arise in the Schur function expansion of the second power of the Vandermonde alternating function, higher powers following by application of the Littlewood-Richardson rule[5]. The Schur functions that arise in the expansion of the second power of the Vandermonde are indexed by ordered partitions, (λ), of the integer $n = N(N-1)$. Di Francesco *etal*[3] defined a class of *admissible partitions*, being those partitions of n associated with non-zero expansion coefficients, c^λ, and determined their number $A(N)$ for up to $N = 29$. They conjectured that these numbers would be exact for every value of N *provided none of the coefficients accidently vanished.* At the time of their conjecture numerical calculations[3,4] supported their conjecture for $N \leq 6$. Scharf *etal*[6] developed algorithms for calculating the expansion coefficents and computed the coefficients for $N = 7, 8, 9$ finding agreement for the conjecture for $N = 7$ but disagreement for $N = 8, 9$. I have recently extended the calculations to $N = 10$.

In this paper I first outline the relationship of the Laughlin wavefunction to the problem of its expansion in terms of signed integers of the even powers of the Vandermonde and define the admissible partitions. In searching for an explanation for the vanishing coefficients I introduce the $q-$discriminant and show that some of the $q-$polynomials have factors that vanish when $q = 1$. Explicit $q-$polynomials are given for the vanishing coefficients when $N = 8$. We then consider properties of the $q-$polynomials and the various possible sums of coefficients which leads to several surprising, though compelling conjectures. Finally we consider, briefly, the relationship of the higher even powers of the Vandermonde and their relationship to Hankel hyperdeterminants.

2. The Laughlin Wavefunction

Laughlin[1] has described the fractional quantum Hall effect in terms of a wavefunction

$$\Psi^m_{Laughlin}(z_1, \ldots, z_N) = \prod_{i<j}^{N} (z_i - z_j)^{2m+1} \exp\left(-\frac{1}{2}\sum_{i=1}^{N} |z_i|^2\right) \tag{1}$$

The Vandermonde alternating function in N variables is defined as

$$V(z_1, \ldots, z_N) = \prod_{i<j}^{N} (z_i - z_j) \tag{2}$$

In terms of the Vandermonde alternating function in N variables, to within an overall normalisation, the Laughlin wavefunction may be written as

$$\frac{\Psi^m_{Laughlin}}{V} = V^{2m} = \sum_{\lambda \vdash n} c^\lambda s_\lambda \quad n = mN(N-1) \tag{3}$$

where the s_λ are Schur functions and the c^λ are signed integers. In most of the following we will discuss the case of $m = 1$. The partitions (λ) indexing the Schur functions are of weight $N(N-1)$. For a given N the partitions are bounded by a highest partition $(2N-2, 2N-4, \ldots, 0)$ and a lowest partition $((N-1)^{N-1})$ with the partitions being of length N and $N-1$.

3. Admissible Partitions

Let

$$n_k = \sum_{i=0}^{k} \lambda_{N-i} - k(k+1) \quad k = 0, 1, \ldots, N-1 \tag{4}$$

Following Di Francesco *et al*[3], we define *Admissible partitions* as satisfying Eq(4) with *all* $n_k \geq 0$. Di Francesco *et al* conjectured that the number of admissible partitions, A_N, was the number of distinct partitions arising in the expansion, Eq(3), *provided none of the coefficients vanished.* The conjecture fails[6] for $N \geq 8$. We find the number of admissible partitions associated with vanishing coefficients as

$$(N = 8) \quad 8, \ (N = 9) \quad 66, \ (N = 10) \quad 389$$

The coefficients of s_λ and s_{λ_r} are equal if[3]

$$(\lambda_r) = (2(N-1) - \lambda_N, \ldots, 2(N-1) - \lambda_1) \tag{5}$$

Such pairs of partitions are said to exhibit *reversal symmetry.* We list the partitions for $N = 8, 9, 10$, having vanishing coefficients, in tables I,II and III respectively.

4. The q−discriminant

Let $q\mathbf{x} = (qx_1, qx_2, \ldots, qx_N)$ and the q-discriminant of $\mathbf{x}$ be

$$D_N(q; \mathbf{x}) = \prod_{1 \leq i \neq j \leq N} (x_i - qx_j) \tag{6}$$

and

$$R_N(q; \mathbf{x}) = \prod_{1 \leq i \neq j \leq N} (x_i - qx_j)(qx_i - x_j) = \sum_\lambda c^\lambda(q) s_\lambda(\mathbf{x}) \tag{7}$$

So that

$$V_N^2(\mathbf{x}) = \prod_{1\le i\ne j\le N}(x_i - x_j)^2 = R_N(1;\mathbf{x}) \tag{8}$$

Introduce q-polynomials such that

$$R_N(q;\mathbf{x}) = \sum_\lambda c^\lambda(q) s_\lambda(\mathbf{x}) \tag{9}$$

$$R_N(q;\mathbf{x}) = \frac{(-1)^{N(N-1)/2}}{(1-q)^N} \sum_{\nu\sqsubseteq(N-1)^N} ((-q)^{|\nu|} + (-q)^{N^2-|\nu|}) s_{(N-1)^N/\nu}(\mathbf{x}) s_{\nu'}(\mathbf{x}) \tag{10}$$

Such expansions have been evaluated as polynomials in q for all admissible partitions for $N = 2, \ldots, 6$ with many examples for $N = 7, 8, 9$. Below we give in Table IV the $q-$polynomials for $N = 2, 3, 4$.

Note that at $N = 4$ some $q-$polynomial factors with some negative coefficients start to appear. As N increases the number of these types of factors grows making possible the "accidental" vanishing of some $q-$polynomials when $q = 1$. This first happens at $N = 8$.

5. The Zero Coefficients and $q-$polynomials

The $q-$polynomials associated with the 8 vanishing coefficients for $N = 8$ have been constructed and are given in Table V. The polynomials are given for each reversal pair of Schur functions. Each $q-$polynomial has a factor $(q-1)^4$ which vanishes for $q = 1$.

6. Sums of Coefficients and Conjectures

Each $q-$polynomial, $P(N,q)$ is of the form

$$P(N,q) = (-1)^\phi q^p Q(q) \tag{11}$$

where ϕ is a phase, p is a positive integer and $Q(q)$ is a polynomial in q. Inspection of Table IV, and many more complex examples, suggests the following conjecture:-

If $N \to N+1$ *then*

$$\phi \to \phi, \quad p \to p + N, \quad Q(q) \to Q(q), \quad \{\lambda\} \to \{2N-2, \lambda\} \tag{12}$$

Di Francesco *et al*[3] give the remarkable result that if

$$V^2(N) = \sum_\lambda c^\lambda s_\lambda$$

then

$$\sum_{\lambda} |c^{\lambda}|^2 = \frac{(3N)!}{N!(3!)^N} \tag{13}$$

Can one write a similar expression for

$$\sum_{\lambda} |c(q)|^2 ? \tag{14}$$

We give below the sums of squares for $N = 2, \ldots, 4$

$$\begin{array}{ll}
N=2 & q^4 + 2q^3 + 4q^2 + 2q + 1 \\
N=3 & q^{12} + 4q^{11} + 11q^{10} + 20q^9 + 34q^8 + 44q^7 \\
 & +52q^6 + 44q^5 + 34q^4 + 20q^3 + 11q^2 + 4q + 1 \\
N=4 & q^{24} + 6q^{23} + 22q^{22} + 58q^{21} + 128q^{20} + 242q^{19} \\
 & +418q^{18} + 646q^{17} + 929q^{16} + 1210q^{15} + 1490q^{14} \\
 & +1670q^{13} + 1760q^{12} + 1670q^{11} + 1490q^{10} + 1210q^9 \\
 & +646q^8 + 418q^6 + 242q^5 + 128q^4 + 58q^3 + 22q^2 + 6q + 1 \\
N=5 & q^{40} + 8q^{39} + 37q^{38} + 124q^{37} + 339q^{36} + 796q^{35} + 1671q^{34} \\
 & +3192q^{33} + 5662q^{32} + 9392q^{31} + 14755q^{30} + 21946q^{29} \\
 & +31190q^{28} + 42202q^{27} + 54902q^{26} + 68238q^{25} + 81835q^{24} \\
 & +93846q^{23} + 104006q^{22} + 110180q^{21} + 112756q^{20} \\
 & +110180q^{19} + 104006q^{18} + 93846q^{17} + 81835q^{16} \\
 & +68238q^{15} + 54902q^{14} + 42202q^{13} + 31190q^{12} \\
 & +21946q^{11} + 14773q^{10} + 9392q^9 + 5662q^8 + 3192q^7 \\
 & +1671q^6 + 796q^5 + 339q^4 + 124q^3 + 37q^2 + 8q + 1
\end{array}$$

We note that in each case the coefficient distribution is symmetric and unimodal.

Consider the sum $CS(N) = \sum_\lambda c^\lambda(1)$

N	No. Partitions	CS(N)
2	2	-2
3	5	-14
4	16	70
5	59	910
6	247	-7280
7	1111	-138320
8	5294	1521520
9	26310	38038000
10	135281	-532532000

From the results tabulated for $N = 2, \ldots, 10$ we conjecture that

$$CS(N) = \prod_{x=0}^{[N/2]} (-3x+1) \prod_{x=0}^{[(N-1)/2]} (6x+1) \tag{15}$$

Again one would like to obtain a $q-$dependent extension of this conjecture.

7. Hankel Determinants and Vandermonde Expansions

The Hankel matrix of order $n+1$ of a sequence $c_0, c_1, \ldots$ is the $n+1$ by $n+1$ matrix whose (i,j) element is c_{i+j} with $0 \leq i, j, \leq n$.

The Hankel determinant of order $n+1$ is the determinant of the corresponding Hankel matrix,

$$det|c_{i+j}|_{0\leq i,j,\leq n} = det \begin{vmatrix} c_0 & c_1 & \ldots & c_n \\ c_1 & c_2 & \ldots & c_{n+1} \\ \vdots & \vdots & \vdots & \\ c_n & c_{n+1} & \ldots & c_{2n} \end{vmatrix} \tag{16}$$

Consider the sequence where $c_n = n!$. Then for $n = 3$ we have the sequence $1, 1, 2, 6, \ldots$

$$D_n^1(c) = det|c_{i+j}|_{0\leq i,j,\leq 3} = det \begin{vmatrix} 1 & 1 & 2 & 6 \\ 1 & 2 & 6 & 24 \\ 2 & 6 & 24 & 120 \\ 6 & 24 & 120 & 720 \end{vmatrix} = 144 \tag{17}$$

We may define a Hankel hyperdeterminant as

$$D_n^k(c) = det_{2k}|c_{i_1+\ldots+i_{2k}}|_{0\leq i_p\leq n-1} \tag{18}$$

NB. The sequence c need not be restricted to just integers but may involve sequences of polynomials etc. Thus Luque and Thibon[7] consider the case where $c_n = h_n(X)$, the $n-th$ complete homogeneous symmetric function of some auxiliary variables $X = \{x_i\}$ and show that $D_n^{(k)}(h)$ may be expressed in terms of Schur functions $s_\lambda(X)$ This problem is equivalent to determining the Schur function expansion of the even powers, Δ^{2k}, of the Vandermonde determinant, Δ!

8. Concluding Remarks

My interest in this subject was stimulated by a meeting with Claude Itzykson at the Formal Power Series and Algebraic Combinatorics meeting held in Florence June 1993. In early 1994 I communicated to him the $N = 8$ counter examples to his conjecture on admissible tableaux. On 1 March 1994 he wrote to me an enthusiastic letter raising many questions for future work. He died on 22 May 1995. This paper discusses some of the questions he raised - but not all - the area seems fruitful with many possible directions remaining to be explored.

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Table I. The partitions (λ) associated with vanishing c^{λ} coefficients for $N = 8$.

$(13\ 11\ 985^{2}41)$	$(13\ 11\ 9854^{2}2)$	$(13\ 11\ 976541)$	$(13\ 10\ 9^{2}6531)$
$(13\ 10\ 987531)$	$(12\ 11\ 97^{2}4^{2}2)$	$(12\ \ 10^{2}96531)$	$(12\ \ 10^{2}7^{2}532)$

Table II. The partitions (λ) associated with vanishing c^{λ} coefficients for $N = 9$.

$(16\ 13\ 11\ 985^{2}41)$	$(16\ 13\ 11\ 9854^{2}2)$	$(16\ 13\ 11\ 976541)$	$(16\ 13\ 10\ 9^{2}6531)$
$(16\ 13\ 10\ 987531)$	$(16\ 12\ 11\ 97^{2}4^{2}2)$	$(16\ 12\ \ 10^{2}96531)$	$(16\ 12\ \ 10^{2}7^{2}532)$
$(15\ 14\ 11\ 985^{2}41)$	$(15\ 14\ 11\ 9854^{2}2)$	$(15\ 14\ 11\ 976541)$	$(15\ 14\ 10\ 9^{2}6531)$
$(15\ 14\ 10\ 987531)$	$(15\ 13\ 11\ 10\ 7^{2}63)$	$(15\ 13\ 11\ 10\ 7^{2}621)$	$(15\ 13\ 11\ 10\ 7^{2}52^{2})$
$(15\ 13\ 11\ 10\ 76^{2}4)$	$(15\ 13\ 11\ 9^{2}652^{2})$	$(15\ 13\ 11\ 9^{2}64^{2}1)$	$(15\ 13\ 11\ 9^{2}6432)$
$(15\ 13\ 11\ 98763)$	$(15\ 13\ 11\ 987621)$	$(15\ 13\ 11\ 976^{2}32)$	$(15\ 13\ 11\ 976542)$
$(15\ 13\ \ 10^{2}85^{2}42)$	$(15\ 13\ 9^{2}7^{2}543)$	$(15\ \ 12^{2}10\ 7^{2}531)$	$(15\ \ 12^{2}9^{2}5^{2}41)$
$(15\ \ 12^{2}9^{2}54^{2}2)$	$(15\ 12\ \ 11^{2}8753)$	$(15\ 12\ \ 11^{2}87521)$	$(15\ 12\ \ 11^{2}7^{2}4^{2}1)$
$(15\ 12\ \ 11^{2}7^{2}432)$	$(15\ 12\ 11\ 10\ 9753)$	$(15\ 12\ 11\ 10\ 97521)$	$(15\ 12\ 11\ 10\ 76^{2}41)$
$(15\ 12\ \ 10^{2}96541)$	$(15\ 10\ 9^{3}5^{4})$	$(14^{2}11\ 10\ 7^{2}531)$	$(14^{2}11\ 9^{2}6531)$
$(14\ 13\ 12\ 10\ 7^{2}531)$	$(14\ 13\ 12\ 9^{2}5^{2}41)$	$(14\ 13\ 12\ 9^{2}54^{2}2)$	$(14\ 13\ 11\ 9^{2}6^{2}4)$
$(14\ 13\ \ 10^{2}97531)$	$(14\ \ 12^{2}11\ 8753)$	$(14\ \ 12^{2}11\ 87521)$	$(14\ \ 12^{2}11\ 7^{2}4^{2}1)$
$(14\ \ 12^{2}11\ 7^{2}432)$	$(14\ \ 12^{2}9^{2}754)$	$(14\ 12\ \ 11^{2}86^{2}31)$	$(14\ 12\ 11\ 10\ 97531)$
$(13^{3}10\ 7643^{2})$	$(13^{2}12\ 10\ 963^{3})$	$(13\ 12\ 11\ 9^{2}7^{2}31)$	$(13\ 12\ \ 10^{2}95^{3}3)$
$(13\ \ 11^{3}76^{2}43)$	$(12^{2}987^{3}64)$	$(12\ \ 10^{3}76^{3}5)$	$(12\ 10\ 9^{3}874^{2})$
$(12\ 10\ 9^{3}85^{3})$	$(11^{4}7^{3}61)$	$(11^{3}976^{3}5)$	$(11^{3}87^{3}64)$
$(11\ \ 10^{3}975^{3})$	$(11\ \ 10^{3}96^{3}4)$		

Table III.The partitions (λ) associated with vanishing c^{λ} coefficients for $N = 10$.

(18 16 13 11 98$5^2$41)	(18 16 13 11 985$4^2$2)	(18 16 13 11 976541)	(18 16 13 10 $9^2$6531)
(18 16 13 10 987531)	(18 16 12 11 9$7^24^2$2)	(18 16 12 $10^2$96531)	(18 16 12 $10^27^2$532)
(18 15 14 11 98$5^2$41)	(18 15 14 11 985$4^2$2)	(18 15 14 11 976541)	(18 15 14 10 $9^2$6531)
(18 15 14 10 987531)	(18 15 13 11 10 $7^2$63)	(18 15 13 11 10 $7^2$621)	(18 15 13 11 10 $7^2$5$2^2$)
(18 15 13 11 10 7$6^2$4)	(18 15 13 11 $9^2$65$2^2$)	(18 15 13 11 $9^2$6$4^2$1)	(18 15 13 11 $9^2$6432)
(18 15 13 11 98763)	(18 15 13 11 987621)	(18 15 13 11 97$6^2$32)	(18 15 13 11 976542)
(18 15 13 $10^2$8$5^2$42)	(18 15 13 $9^27^2$543)	(18 15 $12^2$10 $7^2$531)	(18 15 $12^29^25^2$41)
(18 15 $12^29^2$5$4^2$2)	(18 15 12 $11^2$8753)	(18 15 12 $11^2$87521)	(18 15 12 $11^27^24^2$1)
(18 15 12 $11^27^2$432)	(18 15 12 11 10 9753)	(18 15 12 11 10 97521)	(18 15 12 11 10 7$6^2$41)
(18 15 12 $10^2$96541)	(18 15 10 9^35^4)	(18 $14^2$11 10 $7^2$531)	(18 $14^2$11 $9^2$6531)
(18 14 13 12 10 $7^2$531)	(18 14 13 12 $9^25^2$41)	(18 14 13 12 $9^2$5$4^2$2)	(18 14 13 11 $9^26^2$4)
(18 14 13 $10^2$97531)	(18 14 $12^2$11 8753)	(18 14 $12^2$11 87521)	(18 14 $12^2$11 $7^24^2$1)
(18 14 $12^2$11 $7^2$432)	(18 14 $12^29^2$754)	(18 14 12 $11^2$8$6^2$31)	(18 14 12 11 10 97531)
(18 $13^3$10 764$3^2$)	(18 $13^2$12 10 96$3^3$)	(18 13 12 11 $9^27^2$31)	(18 13 12 $10^2$9$5^3$3)
(18 13 $11^3$7$6^2$43)	(18 $12^2$98$7^3$64)	(18 12 $10^3$7$6^3$5)	(18 12 10 $9^3$87$4^2$)
(18 12 10 $9^3$8$5^3$)	(18 $11^47^3$61)	(18 $11^3$97$6^3$5)	(18 $11^3$8$7^3$64)
(18 11 $10^3$97$5^3$)	(18 11 $10^3$9$6^3$4)	($17^2$13 11 98$5^2$41)	($17^2$13 11 985$4^2$2)
($17^2$13 11 976541)	($17^2$13 10 $9^2$6531)	($17^2$13 10 987531)	($17^2$12 11 9$7^24^2$2)
($17^2$12 $10^2$96531)	($17^2$12 $10^27^2$532)	(17 16 14 11 98$5^2$41)	(17 16 14 11 985$4^2$2)
(17 16 14 11 976541)	(17 16 14 10 $9^2$6531)	(17 16 14 10 987531)	(17 16 13 11 10 $7^2$63)
(17 16 13 11 10 $7^2$621)	(17 16 13 11 10 $7^2$5$2^2$)	(17 16 13 11 10 7$6^2$4)	(17 16 13 11 $9^2$65$2^2$)
(17 16 13 11 $9^2$6$4^2$1)	(17 16 13 11 $9^2$6432)	(17 16 13 11 98763)	(17 16 13 11 987621)
(17 16 13 11 97$6^2$32)	(17 16 13 11 976542)	(17 16 13 $10^2$8$5^2$42)	(17 16 13 $9^27^2$543)
(17 16 $12^2$10 $7^2$531)	(17 16 $12^29^25^2$41)	(17 16 $12^29^2$5$4^2$2)	(17 16 12 $11^2$8753)
(17 16 12 $11^2$87521)	(17 16 12 $11^27^24^2$1)	(17 16 12 $11^27^2$432)	(17 16 12 11 10 9753)
(17 16 12 11 10 97521)	(17 16 12 11 10 7$6^2$41)	(17 16 12 $10^2$96541)	(17 16 10 9^35^4)
(17 $15^2$11 98$5^2$41)	(17 $15^2$11 985$4^2$2)	(17 $15^2$11 976541)	(17 $15^2$10 $9^2$6531)
(17 $15^2$10 987531)	(17 15 14 $11^2$76$3^3$)	(17 15 14 11 $7^3$642)	(17 15 14 11 $7^26^2$43)
(17 15 14 $8^3$7652)	(17 15 14 $8^27^3$43)	(17 15 13 12 11 76$3^3$)	(17 15 13 12 $9^2$852)
(17 15 13 12 $9^2$85$1^2$)	(17 15 13 12 $9^2$843)	(17 15 13 12 $9^2$8421)	(17 15 13 12 $9^2$8$3^2$1)
(17 15 13 12 $9^2$83$2^2$)	(17 15 13 12 $9^2$7$4^2$)	(17 15 13 12 $9^2$7$3^2$2)	(17 15 13 12 $9^25^2$41)
(17 15 13 12 $9^2$5$4^2$2)	(17 15 13 12 9$8^2$62)	(17 15 13 12 9$8^2$6$1^2$)	(17 15 13 12 $8^2$74$3^2$)
(17 15 13 12 876543)	(17 15 13 $11^2$87$4^2$)	(17 15 13 $11^2$87$3^2$2)	(17 15 13 $11^2$8$6^2$3)
(17 15 13 $11^2$8$6^2$21)	(17 15 13 $11^2$8654)	(17 15 13 11 10 9852)	(17 15 13 11 10 985$1^2$)
(17 15 13 11 10 9843)	(17 15 13 11 10 98421)	(17 15 13 11 10 98$3^2$1)	(17 15 13 11 10 983$2^2$)
(17 15 13 11 10 7$5^3$2)	(17 15 13 11 10 7$5^2$43)	(17 15 13 11 9$8^2$54)	(17 15 13 11 98764)
(17 15 13 11 $8^3$6$2^2$)	(17 15 13 11 87$6^2$43)	(17 15 13 $10^2$9$5^3$1)	(17 15 13 $10^2$95$4^2$3)

Table III.The partitions (λ) associated with vanishing c^λ coefficients for $N = 10$.(Contd)

$(17\ 15\ 13\ 10\ 9^{2}5^{3}2)$	$(17\ 15\ 13\ 10\ 9^{2}5^{2}43)$	$(17\ 15\ 13\ 10\ 97^{2}642)$	$(17\ 15\ 13\ 9^{3}7632)$
$(17\ 15\ 13\ 9^{2}875^{2}2)$	$(17\ 15\ \ 12^{3}7652^{2})$	$(17\ 15\ \ 12^{3}764^{2}1)$	$(17\ 15\ \ 12^{3}76432)$
$(17\ 15\ \ 12^{3}75^{2}32)$	$(17\ 15\ \ 12^{2}10\ 7^{2}64)$	$(17\ 15\ 12\ \ 11^{2}75^{3}2)$	$(17\ 15\ 12\ \ 11^{2}75^{2}43)$
$(17\ 15\ 12\ 11\ 97^{2}543)$	$(17\ 15\ 12\ \ 10^{2}8^{2}631)$	$(17\ 15\ \ 11^{2}9^{2}765)$	$(17\ \ 14^{2}13\ 87^{2}43^{2})$
$(17\ \ 14^{2}12\ 11\ 6^{3}31)$	$(17\ \ 14^{2}12\ 11\ 6^{2}541)$	$(17\ \ 14^{2}12\ 11\ 65^{2}3^{2})$	$(17\ \ 14^{2}12\ 9^{2}753)$
$(17\ \ 14^{2}12\ 9^{2}7521)$	$(17\ \ 14^{2}12\ 9^{2}5^{2}41)$	$(17\ \ 14^{2}12\ 9^{2}54^{2}2)$	$(17\ \ 14^{2}\ 11^{2}7^{2}63)$
$(17\ \ 14^{2}\ 11^{2}7^{2}621)$	$(17\ \ 14^{2}\ 11^{2}76^{2}4)$	$(17\ \ 14^{2}10\ 9^{2}7541)$	$(17\ 14\ \ 13^{2}10\ 9752)$
$(17\ 14\ \ 13^{2}10\ 9751^{2})$	$(17\ 14\ \ 13^{2}10\ 9743)$	$(17\ 14\ \ 13^{2}10\ 97421)$	$(17\ 14\ \ 13^{2}10\ 973^{2}1)$
$(17\ 14\ \ 13^{2}10\ 9732^{2})$	$(17\ 14\ \ 13^{2}9^{2}6^{2}3)$	$(17\ 14\ \ 13^{2}9^{2}6^{2}21)$	$(17\ 14\ \ 13^{2}9^{2}654)$
$(17\ 14\ \ 13^{2}9^{2}6531)$	$(17\ 14\ \ 13^{2}9^{2}652^{2})$	$(17\ 14\ \ 13^{2}9^{2}64^{2}1)$	$(17\ 14\ \ 13^{2}9^{2}6432)$
$(17\ 14\ \ 13^{2}9^{2}63^{3})$	$(17\ 14\ \ 13^{2}8^{2}7541)$	$(17\ 14\ 13\ \ 12^{2}7652^{2})$	$(17\ 14\ 13\ \ 12^{2}764^{2}1)$
$(17\ 14\ 13\ \ 12^{2}76432)$	$(17\ 14\ 13\ \ 12^{2}75^{2}32)$	$(17\ 14\ 13\ 12\ 11\ 9752)$	$(17\ 14\ 13\ 12\ 11\ 9751^{2})$
$(17\ 14\ 13\ 12\ 11\ 9743)$	$(17\ 14\ 13\ 12\ 11\ 97421)$	$(17\ 14\ 13\ 12\ 11\ 973^{2}1)$	$(17\ 14\ 13\ 12\ 11\ 9732^{2})$
$(17\ 14\ 13\ 12\ 11\ 76^{2}2^{2})$	$(17\ 14\ 13\ 12\ 11\ 75^{2}42)$	$(17\ 14\ 13\ 12\ 98^{2}63)$	$(17\ 14\ 13\ 12\ 98^{2}621)$
$(17\ 14\ 13\ 12\ 8^{3}62^{2})$	$(17\ 14\ 13\ 12\ 8^{2}6^{2}51)$	$(17\ 14\ 13\ 11\ \ 10^{2}5^{2}41)$	$(17\ 14\ 13\ 11\ \ 10^{2}54^{2}2)$
$(17\ 14\ 13\ 11\ 9^{2}84^{2}1)$	$(17\ 14\ 13\ 11\ 9^{2}8432)$	$(17\ 14\ 13\ 9^{3}84^{2}3)$	$(17\ 14\ \ 12^{2}11\ 8763)$
$(17\ 14\ \ 12^{2}11\ 87621)$	$(17\ 14\ \ 12^{2}11\ 765^{2}1)$	$(17\ 14\ 12\ \ 10^{3}5^{3}2)$	$(17\ 14\ 12\ \ 10^{3}5^{2}43)$
$(17\ 14\ 12\ 10\ 9876^{2}1)$	$(17\ 14\ 11\ \ 10^{3}843^{2})$	$(17\ 14\ 11\ \ 10^{3}6^{2}42)$	$(17\ 14\ 11\ \ 10^{3}5^{3}3)$
$(17\ \ 13^{3}98^{2}531)$	$(17\ \ 13^{2}\ 12^{2}75^{2}3^{2})$	$(17\ \ 13^{2}12\ 11\ 76^{2}41)$	$(17\ 13\ \ 12^{2}\ 10^{2}6541)$
$(17\ 13\ 12\ 11\ 97^{3}43)$	$(17\ \ 12^{2}11\ 10\ 98641)$	$(17\ 12\ \ 11^{3}7^{4})$	$(17\ 12\ \ 10^{5}4^{2}3)$
$(17\ 12\ \ 10^{4}94^{3})$	$(17\ 12\ \ 10^{4}65^{3})$	$(17\ \ 11^{2}10\ 9^{3}83^{2})$	$(16^{2}15\ 11\ 985^{2}41)$
$(16^{2}15\ 11\ 9854^{2}2)$	$(16^{2}15\ 11\ 976541)$	$(16^{2}15\ 10\ 9^{2}6531)$	$(16^{2}15\ 10\ 987531)$
$(16^{2}13\ 12\ 11\ 6^{3}31)$	$(16^{2}13\ 12\ 11\ 6^{2}541)$	$(16^{2}13\ 12\ 11\ 65^{2}3^{2})$	$(16^{2}13\ 12\ 9^{2}753)$
$(16^{2}13\ 12\ 9^{2}7521)$	$(16^{2}13\ 12\ 9^{2}5^{2}41)$	$(16^{2}13\ 12\ 9^{2}54^{2}2)$	$(16^{2}13\ \ 11^{2}8753)$
$(16^{2}13\ \ 11^{2}87521)$	$(16^{2}\ 12^{2}11\ 76541)$	$(16^{2}12\ \ 10^{3}7531)$	$(16^{2}12\ \ 10^{3}6541)$
$(16^{2}11\ \ 10^{3}5^{3}2)$	$(16^{2}11\ \ 10^{3}5^{2}43)$	$(16\ \ 15^{2}11\ 10\ 7^{2}531)$	$(16\ \ 15^{2}11\ 9^{2}6531)$
$(16\ 15\ 14\ 13\ 87^{2}43^{2})$	$(16\ 15\ 14\ 12\ 11\ 6^{3}31)$	$(16\ 15\ 14\ 12\ 11\ 6^{2}541)$	$(16\ 15\ 14\ 12\ 11\ 65^{2}3^{2})$
$(16\ 15\ 14\ 12\ 9^{2}753)$	$(16\ 15\ 14\ 12\ 9^{2}7521)$	$(16\ 15\ 14\ 12\ 9^{2}5^{2}41)$	$(16\ 15\ 14\ 12\ 9^{2}54^{2}2)$
$(16\ 15\ 14\ \ 11^{2}7^{2}63)$	$(16\ 15\ 14\ \ 11^{2}7^{2}621)$	$(16\ 15\ 14\ \ 11^{2}76^{2}4)$	$(16\ 15\ 14\ 10\ 9^{2}7541)$
$(16\ 15\ \ 13^{2}11\ 6^{3}31)$	$(16\ 15\ \ 13^{2}11\ 6^{2}541)$	$(16\ 15\ \ 13^{2}11\ 65^{2}3^{2})$	$(16\ 15\ 13\ \ 11^{2}8^{2}62)$
$(16\ 15\ 13\ \ 11^{2}8^{2}61^{2})$	$(16\ 15\ \ 12^{2}11\ 9753)$	$(16\ 15\ \ 12^{2}11\ 97521)$	$(16\ 15\ 12\ 11\ 9^{3}531)$
$(16\ 15\ 12\ \ 10^{2}8763^{2})$	$(16\ \ 14^{2}13\ 10\ 9752)$	$(16\ \ 14^{2}13\ 10\ 9751^{2})$	$(16\ \ 14^{2}13\ 10\ 9743)$
$(16\ \ 14^{2}13\ 10\ 97421)$	$(16\ \ 14^{2}13\ 10\ 973^{2}1)$	$(16\ \ 14^{2}13\ 10\ 9732^{2})$	$(16\ \ 14^{2}13\ 9^{2}6^{2}3)$
$(16\ \ 14^{2}13\ 9^{2}6^{2}21)$	$(16\ \ 14^{2}13\ 9^{2}654)$	$(16\ \ 14^{2}13\ 9^{2}6531)$	$(16\ \ 14^{2}13\ 9^{2}652^{2})$

Table III.The partitions (λ) associated with vanishing c^λ coefficients for $N = 10$.(Contd)

$(16\ \ 14^2 13\ 9^2 64^2 1)$	$(16\ \ 14^2 13\ 9^2 6432)$	$(16\ \ 14^2 13\ 9^2 63^3)$	$(16\ \ 14^2 13\ 8^2 7541)$
$(16\ \ 14^2\ 11^2 9762)$	$(16\ \ 14^2\ 11^2 9761^2)$	$(16\ \ 14^2 11\ 987542)$	$(16\ 14\ \ 13^2 11\ 76541)$
$(16\ 14\ \ 13^2 10\ 8^2 53)$	$(16\ 14\ \ 13^2 10\ 8^2 521)$	$(16\ 14\ 13\ 12\ 11\ 9753)$	$(16\ 14\ 13\ 12\ 11\ 97521)$
$(16\ 14\ 13\ 11\ 10\ 974^2 2)$	$(16\ 14\ \ 12^2 8^3 741)$	$(16\ 14\ 12\ \ 11^3 7431)$	$(16\ 14\ 12\ \ 11^2 98531)$
$(16\ 14\ 12\ 11\ 987652)$	$(16\ \ 13^3 12\ 75^2 3^2)$	$(16\ \ 13^3 11\ 87531)$	$(16\ \ 13^3 11\ 7^2 631)$
$(16\ \ 13^3 9^2 8531)$	$(16\ \ 13^3 8^3 72^2)$	$(16\ \ 13^3 8^3 641)$	$(16\ \ 13^2 11\ 10\ 9^2 531)$
$(16\ 13\ \ 12^2 11\ 76^2 43)$	$(16\ 13\ 12\ 11\ \ 10^3 431)$	$(16\ 13\ 12\ 11\ 10\ 97642)$	$(16\ \ 11^2 10\ 987^3 4)$
$(16\ \ 11^2 987^5)$	$(16\ 11\ \ 10^5 54^2)$	$(15^3 12\ 11\ 7^2 431)$	$(15^3 12\ 11\ 76531)$
$(15^3 12\ 9^2 5^2 41)$	$(15^3 12\ 9^2 54^2 2)$	$(15^3 12\ 9865^2)$	$(15^2 14\ 12\ 11\ 85^3)$
$(15^2 14\ 12\ 11\ 84^3 3)$	$(15^2 14\ \ 11^2 10\ 54^2 1)$	$(15^2 14\ \ 11^2 10\ 5432)$	$(15^2 14\ 11\ \ 10^2 6531)$
$(15^2 14\ 10\ 8^3 741)$	$(15^2\ 13^2 12\ 7652^2)$	$(15^2\ 13^2 12\ 764^2 1)$	$(15^2\ 13^2 12\ 76432)$
$(15^2\ 13^2 12\ 75^2 32)$	$(15^2\ 13^2 11\ 6^2 5^2 1)$	$(15^2\ 13^2 11\ 65^3 2)$	$(15^2\ 13^2 11\ 65^2 43)$
$(15^2 12\ 11\ 10\ 8^2 632)$	$(15^2 10\ 9^3 87^2 1)$	$(15\ \ 14^3 10\ 7643^2)$	$(15\ \ 14^2 13\ 98^2 531)$
$(15\ \ 14^2 10\ 9^3 541)$	$(15\ \ 14^2 8^5 61)$	$(15\ 14\ \ 13^2 12\ 75^2 3^2)$	$(15\ 14\ \ 13^2 11\ 87531)$
$(15\ 14\ \ 13^2 11\ 7^2 631)$	$(15\ 14\ \ 13^2 9^2 8531)$	$(15\ 14\ \ 13^2 8^3 72^2)$	$(15\ 14\ \ 13^2 8^3 641)$
$(15\ 14\ 13\ 12\ 11\ 10\ 6531)$	$(15\ 14\ 13\ \ 11^2 9^2 53)$	$(15\ 14\ 13\ \ 11^2 9^2 521)$	$(15\ 14\ 13\ \ 11^2 97631)$
$(15\ 14\ \ 12^2\ 11^2 7431)$	$(15\ 14\ \ 12^2 11\ 10\ 7531)$	$(15\ 14\ \ 12^2 11\ 7^3 5)$	$(15\ 14\ \ 12^2 11\ 76^2 52)$
$(15\ 14\ \ 12^2 11\ 765^2 3)$	$(15\ 14\ \ 11^3\ 10^2 431)$	$(15\ 14\ \ 11^3 97651)$	$(15\ \ 13^3 98^2 65)$
$(15\ \ 13^3 8^3 741)$	$(15\ \ 13^2 12\ 11\ 76^2 43)$	$(15\ \ 12^2 10\ 8^2 7^2 65)$	$(15\ 12\ 11\ 9^2 8^2 7^2 4)$
$(15\ \ 11^2 10\ 9^2 7^3 4)$	$(15\ 11\ \ 10^3 98764)$	$(14^3 98^4 61)$	$(14^2 13\ 8^5 72)$
$(14^2 13\ 8^2 7^4 5)$	$(14^2 11\ 10\ 9^3 86)$	$(14\ \ 12^3 98^3 7)$	$(14\ 12\ \ 11^3 10\ 96^2)$
$(14\ 12\ \ 11^3 10\ 7^3)$	$(14\ 12\ 11\ 10\ 98^3 73)$	$(14\ \ 11^3 10\ 987^2 2)$	$(14\ \ 11^3 9^2 87^2 3)$
$(14\ \ 11^2\ 10^2 9^2 763)$	$(14\ \ 11^2 10\ 9^2 7^3 5)$	$(14\ 11\ \ 10^2 8^3 7^3)$	$(13^4 9^3 83)$
$(13^4 9^3 821)$	$(13^3 12\ 8^4 61)$	$(13^3 11\ 98^3 7)$	$(13^3 10\ 9^3 86)$
$(13\ \ 12^3 11\ 97^3)$	$(13\ \ 12^3 11\ 8^3 6)$	$(13\ 12\ \ 11^2\ 10^2 86^2 3)$	$(13\ 12\ 11\ 8^5 7^2)$
$(13\ \ 11^4\ 10^2 54^2)$	$(13\ \ 11^3 9^2 87^2 4)$	$(11^5 10\ 97^2 2)$	$(11^3\ 10^3 8^2 74)$
$(11^2\ 10^5 765)$			

Table IV. The $q-$polynomials for $N = 2, 3, 4$. The square brackets encase the signed coefficients, $c^{\lambda}(1)$, for $q = 1$, the next column the $q-$polynomial and the last column the associated Schur functions, $\{\lambda\}$.

N = 2

[1]	q	$\{2\}$
[−3]	$-(q^2+q+1)$	$\{1^2\}$

N = 3

[1]	q^3	$\{42\}$
[−3]	$-q^2(q^2+q+1)$	$\{41^2\}+\{3^2\}$
[6]	$+q(q^2+q+1)(q^2+1)$	$\{321\}$
[−15]	$-(q^2+q+1)(q^4+q^3+q^2+q+1)$	$\{2^3\}$

N = 4

[1]	q^6	$\{642\}$
[−3]	$-q^5(q^2+q+1)$	$\{641^2\}+\{63^2\}+\{5^22\}$
[6]	$+q^4(q^2+1)(q^2+q+1)$	$\{6321\}+\{543\}$
[9]	$+q^4(q^2+q+1)^2$	$\{5^21^2\}$
[−15]	$-q^3(q^2+q+1)(q^4+q^3+q^2+q+1)$	$\{62^3\}+\{4^3\}$
[−12]	$-q^3(q^2+q+1)(q^2+1)^2$	$\{5421\}$
[−9]	$-q^3(q^2-q+1)(q^2+q+1)^2$	$\{53^21\}$
[−6]	$-q^3(q^2+q+1)(q^4+1)$	$\{4^22^2\}$
[27]	$+q^2(q^2-q+1)(q^2+q+1)^3$	$\{532^2\}+\{4^231\}$
[−45]	$-q(q^2-q+1)(q^4+q^3+q^2+q+1)(q^2+q+1)^2$	$\{43^22\}$
[105]	$+(q^2+q+1)(q^4+q^3+q^2+q+1)$	$\{3^4\}$
	$\times(q^6+q^5+q^4+q^3+q^2+q+1)$	

Table IV. The $q-$polynomials for $N = 2, 3, 4$. The square brackets encase the signed coefficients, $c^{\lambda}(1)$, for $q = 1$, the next column the $q-$polynomial and the last column the associated Schur functions, $\{\lambda\}$. (Contd)

N = 5

[1]	q^{10}	$\{8642\}$
[−3]	$-q^9(q^2+q+1)$	$\{86412^2\} + \{8632^2\} + \{85^22\} + \{7^242\}$
[6]	$+q^8(q^2+1)(q^2+q+1)$	$\{86321\} + \{8543\} + \{7652\}$
[9]	$+q^8(q^2+q+1)^2$	$\{85^21^2\} + \{7^241^2\} + \{7^23^2\}$
[−12]	$-q^7(q^2+q+1)(q^2+1)^2$	$\{85421\} + \{7643\}$
[−9]	$-q^7(q^2-q+1)(q^2+q+1)^2$	$\{853^21\} + \{75^23\}$
[−6]	$-q^7(q^2+q+1)(q^4+1)$	$\{84^22^2\} + \{6^24^2\}$
[−15]	$-q^7(q^2+q+1)(q^4+q^3+q^2+q+1)$	$\{862^3\} + \{84^3\} + \{6^32\}$
[−18]	$-q^7(q^2+1)(q^2+q+1)^2$	$\{7^2321\} + \{7651^2\}$
[27]	$+q^6(q^2-q+1)(q^2+q+1)^3$	$\{8532^2\} + \{84^231\} + \{754^2\} + \{6^253\}$
[24]	$+q^6(q^2+q+1)(q^2+1)^3$	$\{76421\}$
[18]	$+q^6(q^2+1)(q^2-q+1)(q^2+q+1)^2$	$\{763^21\} + \{75^221\}$
[45]	$+q^6(q^4+q^3+q^2+q+1)(q^2+q+1)^2$	$\{7^22^3\} + \{6^31^2\}$
[−45]	$-q^5(q^2-q+1)(q^4+q^3+q^2+q+1)$	$\{843^22\} + \{65^24\}$
	$\times(q^2+q+1)^2$	
[−54]	$-q^5(q^2+1)(q^2-q+1)(q^2+q+1)^3$	$\{7632^2\} + \{6^2521\}$
[−36]	$-q^5(q^2-q+1)(q^2+q+1)^2(q^2+1)^2$	$\{75431\} + \{74^31\}$
[−27]	$-q^5(q^2-q+1)^2(q^2+q+1)^3$	$\{7542^2\} + \{6^2431\}$
[−18]	$-q^5(q^2-q+1)(q^4+1)(q^2+q+1)^2$	$\{6^23^22\} + \{65^22^2\}$
[105]	$+q^4(q^2+q+1)(q^4+q^3+q^2+q+1)$	$\{83^4\} + \{5^4\}$
	$\times(q^6+q^5+q^4+q^3+q^2+q+1)$	
[81]	$+q^4(q^2-q+1)^2(q^2+q+1)^4$	$(\{753^22\} + \{65^231\}$
[72]	$+q^4(q^4+1)(q^2+1)^2(q^2+q+1)^2$	$(\{74^232\} + \{654^21\}$
[111]	$+q^4(q^2+q+1)(q^{10}+2q^9+4q^8+3q^7$	$\{6^242^2\}$
	$+6q^6+5q^5+6q^4+3q^3+4q^2+2q+1)$	
[45]	$+q^4(q^4-q^3+q^2-q+1)(q^4+q^3+q^2+q+1)$	$\{653^3\} + \{5^332\}$
	$\times(q^2+q+1)^2$	
[−180]	$-q^3(q^2+1)(q^4+q^3+q^2+q+1)(q^4+1)$	$\{743^3\} + \{5^341\}$
	$\times(q^2+q+1)^2$	
[−144]	$-q^3(q^4+1)(q^2+q+1)^2(q^2+1)^3$	$\{65432\}$
[−90]	$-q^3(q^4+q^3+q^2+q+1)(q^2-q+1)(q^4+1)$	$\{64^32\}$
	$(q^2+q+1)^2$	
[−75]	$-q^3(q^2+q+1)(q^4-q^3+q^2-q+1)$	$\{5^243^2\}$
	$\times(q^4+q^3+q^2+q+1)^2$	
[270]	$+q^2(q^4+q^3+q^2+q+1)(q^2-q+1)(q^4+1)$	$\{64^23^2\} + \{5^24^22\}$
	$(q^2+q+1)^3$	
[−420]	$-q(q^2+q+1)(q^2+1)(q^4+q^3+q^2+q+1)(q^4+1)$	$\{54^33\}$
	$\times(q^6+q^5+q^4+q^3+q^2+q+1)$	
[945]	$+(q^4+q^3+q^2+q+1)(q^6+q^3+1)$	$\{4^5\}$
	$\times(q^2+q+1)^2(q^6+q^5+q^4+q^3+q^2+q+1)$	

Table V. The $q-$polynomials for the eight vanishing coefficients, $c^{\lambda}(1)$, for $N=8$.

$-q^{17}(q^2-q+1)^2(q^2+1)^2(q-1)^4(q^2+q+1)^5$	$\{13\ 11985^2 41\}, \{13\ 10\ 9^2 6531\}$
$+q^{16}(q^2+1)(q^2-q+1)^3(q-1)^4(q^2+q+1)^6$	$\{13\ 11\ 9854^2 2\}, \{13\ 10\ 987531\}$
$+q^{16}(q^2-q+1)^2(q^2+1)^3(q-1)^4(q^2+q+1)^5$	$\{13\ 11\ 976541\}, \{12\ 10^2\ 96531\}$
$+q^{14}(q^{10}+q^9+3q^8+4q^6+q^5+4q^4+3q^2+q+1)$ $\times(q^2-q+1)^2(q-1)^4(q^2+q+1)^5$	$\{12\ 11\ 97^2 4^2 2\}, \{12\ 10^2\ 7^2 532\}$