Generating functions for stable branching coefficients of $U(n) \downarrow S_n$, $O(n) \downarrow S_n$ and $O(n-1) \downarrow S_n$

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Abstract. We propose an algorithm computing explicit generating functions of the stable multiplicities of irreducible representations $(n - |\mu|, \mu)$ of S_n arising in the restriction from $U(n)$, $O(n)$ or $O(n-1)$ to S_n of an irreducible tensor representation of the unitary or orthogonal group; i.e., we compute the multiplicities in a way which is independent of n and m, the weight of the label $(m - |\lambda|, \lambda)$ of the corresponding irrep.

1. Introduction

The symmetric group S_n plays an important role in those areas of physics and chemistry involving permutational symmetry such as in the implementation of the Pauli exclusion principle in constructing totally antisymmetric wave functions for identical fermions. Such applications arise, for example, in the classification of $n-$electron states and in symplectic models of nuclei and quantum dots [4]. These applications frequently require the resolution of symmetrised powers of the irreducible tensor representations of S_n. This situation arises in the case of evaluating the branching coefficients for the group-subgroup restrictions $U(n) \downarrow S_n$, $O(n) \downarrow S_n$ and $O(n-1) \downarrow S_n$. In these cases the coefficients involve the inner plethysms [11, 13]:

$$U(n) \downarrow S_n \qquad \{\lambda\} \downarrow \langle 1 \rangle \otimes \{\lambda/M\} \tag{1}$$

$$O(n) \downarrow S_n \qquad [\lambda] \downarrow \langle 1 \rangle \otimes \{\lambda/G\} \tag{2}$$

$$O(n-1) \downarrow S_n \qquad [\lambda] \downarrow \langle 1 \rangle \otimes \{\lambda/C\} \tag{3}$$

where M, C and G are infinite series of $S-$functions [6, 5] and the reduced notation for S_n [10, 7, 8, 14] is exploited.

The evaluation of the above inner plethysms is the key problem considered herein. Most previous formulations have involved Littlewood's methods [6, 7, 12], see also [1, 2] for the special cases $\lambda = (n), (1^n)$, for obtaining the complete content of the plethysm whereas interest often lies in the computation of specific coefficients. To that end it is highly desirable to be able to construct generating functions to yield the multiplicities of irreducible representations $(n - |\mu|, \mu)$ of S_n. The classical methods allow to give n-independent results, e.g. for the restriction of an irrep $\{\lambda\}$ of $U(n)$ to S_n. Here we show that it is also possible to obtain branching formulae which are also independent of the greatest part λ_1 of λ. We construct explicit algorithms for obtaining the relevant generating functions using the formalism of vertex operators in the same way as in [14, 3, 13] and demonstrate the procedure with several illustrative examples.

2. Notations and Background

Our standard reference for symmetric functions will be [9] and we will adopt its notations as in [13]. A Schur function is denoted by $\{\lambda\}$ or by s_λ, depending on whether it is interpreted as a character of $U(n)$ or as an operator. The inner plethysm $(\lambda) \otimes \{\mu\}$ is denoted either by $\hat{s}_\mu(s_\lambda)$ or by $s_\mu {}^\wedge s_\lambda$. We recall that symmetric functions of a formal difference of variable sets are defined by $p_k(X - Y) = p_k(X) - p_k(Y)$ and that the set $\{1, z, z^2, \ldots\}$ is identified with the power series $(1-z)^{-1}$.

An algebraic formulation of Littlewood's reduced notation for symmetric functions can be given by considering a particular case of a vertex operator

$$\Gamma_z s_\lambda := \sum_{n\in\mathbf{Z}} s_{(n,\lambda)} z^n.$$

That is, Littlewood's reduced notation $\langle\lambda\rangle$ has to be interpreted as the infinite series $\Gamma_1 s_\lambda$. We sketch some of its basic properties and refer the reader to [14, 3, 13], e.g., for details (see also [9], in particular ex.29, p.95ff., ex.3, p.75f., ex.25, p.91ff).
The adjoints (w.r.t. Hall's inner product) of multiplication of symmetric functions by

$$\sigma_z(X) := \sum_{n\geq 0} h_n(X) z^n, \text{ and } \lambda_z(X) := \sum_{n\geq 0} e_n(X) z^n,$$

the generating functions for the complete symmetric and elementary symmetric functions, are algebra automorphisms $D_{\sigma_z}, D_{\lambda_z}$ and we have

$$D_{\sigma_z} \circ D_{\lambda_{-z}} = id.$$

In Lambda-ring notation these operators can be described as

$$D_{\sigma_z} F(X) = F(X+z), \qquad D_{\lambda_{-z}} F(X) = F(X-z),$$

for any symmetric function F. This amounts to say that for the power sums we have

$$D_{\sigma_z} p_k(X) = p_k(X) + z^k, \qquad D_{\lambda_{-z}} p_k(X) = p_k(X) - z^k,$$

and for Schur functions

$$D_{\sigma_z} s_\lambda(X) = \sum_{i\geq 0} s_{\lambda/(i)}(X) z^i, \qquad D_{\lambda_{-z}} s_\lambda(X) = \sum_{i\geq 0} s_{\lambda/(1^i)}(X)(-z)^i.$$

Finally, the vertex operator has a factorization

$$\Gamma_z F(X) = \sigma_z(X) D_{\lambda_{-1/z}} F(X) = \sigma_z(X) F(X - 1/z) \tag{4}$$

a formula, which we will use in the sequel.

3. Stable inner plethysm multiplicities

3.1. The result

In this section, we address the problem of computing in an n and m independent way the multiplicity of an irreducible representation $(n-|\mu|,\mu)$ of S_n in the inner plethysm

$\langle 1\rangle \otimes \{m-|\lambda|, \lambda\}$. This will be applied to the calculation of the branching coefficients of (1)-(3) in the forthcoming sections.

We know that Littlewood's reduced notation allows us to expand the inner plethysm $\langle 1\rangle \otimes \{\nu\}$ as a linear combination of stable characters:

$$\langle 1\rangle \otimes \{\nu\} = \sum_{\mu} d_\mu \, \langle \mu\rangle \tag{5}$$

so that for any n such that $n-|\mu| \geq \mu_1$, the multiplicity of the irrep $(n-|\mu|, \mu)$ of S_n will always be equal to d_μ.

Here, we go one step further, and allow the first part ν_1 of ν to be arbitrary. That is, we set $\nu = (\nu_1, \lambda)$, and we consider the generating function

$$F_\lambda(z) := \sum_{n\in\mathbf{Z}} \langle 1\rangle \otimes \{n, \lambda\} z^n := \sum_{n\in\mathbf{Z}} z^n \, \hat{s}_{(n,\lambda)}\langle 1\rangle := \sum_{\mu} c_{\lambda\mu}(z)\langle\mu\rangle. \tag{6}$$

Then, the $c_{\lambda\mu}(z)\varphi(z)$, where $\varphi(z) = \prod_{k\geq 1}(1-z^k)$, are rational functions which can be explicitly computed by the procedure described below.

For example, with $\lambda = (2,1)$, we obtain $c_{\lambda\mu}(z) = (1-z)\varphi(z)^{-1} a_{\lambda\mu}(z)$, with

$$a_{21,0}(z) = \frac{z^4}{(1-z^3)\,(1-z)^2} \tag{7}$$

$$a_{21,1}(z) = \frac{1-3z+2z^2+3z^4-2z^5-z^6+z^7}{z^2\,(1-z)^3\,(1-z^3)} \tag{8}$$

$$a_{21,2}(z) = \frac{(1+z-z^3)\,z^2}{(1-z)^4\,(1-z^3)\,(1+z)} \tag{9}$$

$$a_{21,11}(z) = \frac{(1+z-z^3-z^4+z^5)z^2}{(1-z)^3(1-z^2)(1-z^3)} \tag{10}$$

and for $\mu = (4,2,2)$, $a_{21,422}(z)$ is equal to

$$\frac{z^6\,(z^{16}-z^{14}-2\,z^{13}-2\,z^{12}+z^{11}+z^{10}+3\,z^9+z^8+z^7+2\,z^6-z^4-3\,z^3-z^2-1)}{(z^6-1)\,(z+z^2+1)\,(z-1)^6\,(z^2-1)^3\,(z^3-1)\,(z^4+z^3+z^2+z+1)} \tag{11}$$

Taking the Taylor expansion of the latter up to order 10, we get that the multiplicity of $(30,4,2,2)$ in the inner plethysm $(37,1)\otimes\{10,2,1\}$ is equal to 125.

Similarly we find that the multiplicity of $(31,3,3,2,1)$ in $(39,1)\otimes\{30,4,3,2,1\}$ is equal to 309727790880.

3.2. Derivation of the generating functions

Using the vertex operator formula (4)

$$\sum_{n\in\mathbf{Z}} z^n \, s_{(n,\lambda)} = \Gamma_z s_\lambda = \sigma_z(X) s_\lambda \left(X - \frac{1}{z}\right) \tag{12}$$

and taking into account the inner plethysm series (cf. [1])

$$\hat{\sigma}_z\langle 1\rangle = \sum_{n\geq 0} \hat{h}_n\langle 1\rangle z^n = (1-z)\sigma_1\left(\frac{X}{1-z}\right) = (1-z)\prod_{k\geq 0}\left(\sum_{m\geq 0} z^{km}h_m\right) \quad (13)$$

we have

$$\begin{aligned} F_\lambda(z) &= \left[\sigma_z(X)s_\lambda\left(X-\frac{1}{z}\right)\right]^\wedge \langle 1\rangle \\ &= \hat{\sigma}_z\langle 1\rangle \ * \ s_\lambda\left(X-\frac{1}{z}\right)^\wedge \langle 1\rangle \\ &= (1-z)\sigma_1\left(\frac{X}{1-z}\right) \ * \ U, \end{aligned} \quad (14)$$

where $U := [s_\lambda(X-\frac{1}{z})]^\wedge\langle 1\rangle$.
For example, with $\lambda = (2,1)$ one would have

$$U = \left(s_{21} - \frac{1}{z}s_{21/1} + \frac{1}{z^2}s_{21/1^2}\right)^\wedge \langle 1\rangle = \hat{s}_{21}\langle 1\rangle - \frac{1}{z}(\langle 1\rangle * \langle 1\rangle) + \frac{1}{z^2}\langle 1\rangle \ .$$

To evaluate such an expression, we use Littlewood's formula, which gives the result as a combination of stable characters $\langle\mu\rangle = \sigma_1 D_{\lambda_{-1}} s_\mu = \sigma_1 \cdot s_\mu(X-1)$, and we keep apart the factor σ_1, writing $U = \sigma_1 \cdot H$.
In our example, we obtain

$$\hat{s}_{21}\langle 1\rangle = \langle 1\rangle \otimes \{21\} = \langle 21\rangle + \langle 2\rangle + \langle 11\rangle + \langle 1\rangle = \sigma_1 \cdot s_{21} \quad (15)$$

$$\hat{s}_{21/1}\langle 1\rangle = \langle 1\rangle \ * \ \langle 1\rangle = \langle 2\rangle + \langle 11\rangle + \langle 1\rangle + \langle 0\rangle = \sigma_1 \cdot (s_2 + s_{11} - s_1 + 1) \quad (16)$$

$$\hat{s}_{21/11}\langle 1\rangle = \langle 1\rangle = \sigma_1 \cdot (s_1 - 1) \quad (17)$$

and, finally,

$$U = \sigma_1 \cdot \left(s_{21} - \frac{1}{z}(s_2 + s_{11} - s_1 + 1) + \frac{1}{z^2}(s_1 - 1)\right) = \sigma_1 \cdot H \quad (18)$$

Next, we expand the internal product (14), taking into account the property

$$\sigma_1\left(\frac{X}{1-z}\right) * F(X) = F\left(\frac{X}{1-z}\right)$$

which gives

$$\begin{aligned} F_\lambda(z) &= (1-z)\,\sigma_1\left(\frac{X}{1-z}\right) * U(X) \\ &= (1-z)\,U\left(\frac{X}{1-z}\right) \\ &= (1-z)\,\sigma_1\left(\frac{X}{1-z}\right)\ H\left(\frac{X}{1-z}\right). \end{aligned} \quad (19)$$

Now we extract a vertex operator by writing

$$\sigma_1\left(\frac{X}{1-z}\right) = \sigma_1(X) \cdot \sigma_1\left(\frac{zX}{1-z}\right)$$

and (note that $D_{\lambda_{-1}} \circ D_{\sigma_1} = \mathrm{id}$)

$$\begin{aligned}\sigma_1\left(\frac{zX}{1-z}\right) H\left(\frac{X}{1-z}\right) &= D_{\lambda_{-1}}\, \sigma_1\left(\frac{z(X+1)}{1-z}\right) H\left(\frac{X+1}{1-z}\right) \\ &= \sigma_1\left(\frac{z}{1-z}\right) D_{\lambda_{-1}}\left[\sigma_1\left(\frac{zX}{1-z}\right) H\left(\frac{X+1}{1-z}\right)\right].\end{aligned}$$

Thus,

$$\begin{aligned}F_\lambda(z) &= (1-z)\,\sigma_1\left(\frac{z}{1-z}\right) \Gamma_1\left[\sigma_1\left(\frac{zX}{1-z}\right) H\left(\frac{X+1}{1-z}\right)\right] \\ &= \prod_{i\geq 2}\frac{1}{1-z^i}\,\Gamma_1\left[\sigma_1\left(\frac{zX}{1-z}\right) H\left(\frac{X+1}{1-z}\right)\right]. \qquad (20)\end{aligned}$$

If we write

$$\sigma_1\left(\frac{zX}{1-z}\right) H\left(\frac{X+1}{1-z}\right) = \sum_\mu a_{\lambda\mu}(z) s_\mu,$$

then

$$F_\lambda(z) = \sum_\mu c_{\lambda\mu}(z)\langle\mu\rangle$$

with

$$c_{\lambda\mu}(z) = a_{\lambda\mu}(z)\prod_{i\geq 2}\frac{1}{1-z^i}.$$

We can now compute $a_{\lambda\mu}(z)$ using properties of the scalar product and adjoint operators:

$$\begin{aligned}a_{\lambda\mu}(z) &= \left\langle s_\mu\ ,\ \sigma_1\left(\frac{zX}{1-z}\right) H\left(\frac{X+1}{1-z}\right)\right\rangle \\ &= \left\langle s_\mu\ ,\ \prod_{k\geq 1}\sigma_{z^k}(X)\ \cdot\ H\left(\frac{X+1}{1-z}\right)\right\rangle \\ &= \left\langle s_\mu\left(X+\frac{z}{1-z}\right)\ ,\ H\left(\frac{X+1}{1-z}\right)\right\rangle. \qquad (21)\end{aligned}$$

Now we expand $H(\frac{X+1}{1-z})$ by replacing each power sum p_k in the expansion of H by $(p_k+1)/(1-z^k)$. Let the rational functions $d_{\lambda\alpha}(z)$ be defined by

$$H\left(\frac{X+1}{1-z}\right) = \sum_\alpha d_{\lambda\alpha}(z) s_\alpha(X). \qquad (22)$$

Now,

$$s_\mu\left(X+\frac{z}{1-z}\right) = \sum_{\alpha\subseteq\mu} s_\alpha(X) s_{\mu/\alpha}\left(\frac{z}{1-z}\right)\ ,$$

so that

$$a_{\lambda\mu}(z) = \sum_\alpha d_{\lambda\alpha}(z) s_{\mu/\alpha}\left(\frac{z}{1-z}\right)$$

For $\lambda = (2,1)$, the coefficients $d_{21,\alpha}(z)$ are given by

$$\begin{aligned} H\left(\frac{X+1}{1-z}\right) &= \frac{z}{(1-z^3)(z-1)^2}\, s_3 + \frac{1+z^2}{(1-z^3)(z-1)^2}\, s_{21} + \frac{z}{(1-z^3)(z-1)^2}\, s_{111} \\ &+ \frac{2\,z-1}{z(1-z)^3}\, s_2 + \frac{2\,z-1}{z(1-z)^3}\, s_{11} + \frac{1-3z+2z^2+z^3}{z^2(1-z)^3}\, s_1 + \frac{z^4}{(1-z^3)(z-1)^2} s_0 \end{aligned} \tag{23}$$

From these expressions, we obtain the required generating functions.

3.3. Summary of the algorithm

To compute the generating function $c_{\lambda\mu}(z)$:

(i) Evaluate $f = s_\lambda(X - \frac{1}{z})$, either by expanding s_λ on the basis p_α and replacing each power sum p_k by $p_k - z^{-k}$, or by the more efficient formula

$$s_\lambda\left(X - \frac{1}{z}\right) = \sum_{r=0}^{\ell(\lambda)} \left(-\frac{1}{z}\right)^r s_{\lambda/1^r}(X)\ .$$

(ii) Compute $U = \hat{f}\langle 1\rangle$ as a linear combination of stable characters $\langle\mu\rangle$ by means of Littlewood's formula, and write it in the form $U = \sigma_1 \cdot H$, taking into account the fact that $\langle\mu\rangle = \sigma_1 \cdot s_\mu(X-1)$.

(iii) Evaluate $H\left(\frac{X+1}{1-z}\right)$, for example by expanding H in terms of power sums, and replacing each p_k by $(p_k+1)/(1-z^k)$.

(iv) Take the scalar product of the previous expression with

$$s_\mu\left(X + \frac{z}{1-z}\right) = \sum_{\nu\subseteq\mu} s_\nu\left(\frac{z}{1-z}\right) s_{\mu/\nu}(X)$$

(a closed formula for $s_\nu\left(\frac{z}{1-z}\right)$ can be found for example in [9], ex. 2 p. 45). This yields $a_{\lambda\mu}(z)$.

(v) $c_{\lambda\mu}(z) = (1-z)\varphi(z)^{-1} a_{\lambda\mu}(z)$.

The multiplicity of the irreducible representation $(n-|\mu|,\mu)$ of S_n (for any n) in the inner plethysm $\langle 1\rangle \otimes \{m,\lambda\}$ is then equal to the coefficient of z^m in the Taylor expansion of $c_{\lambda\mu}(z)$.

4. The restriction $U(n) \downarrow S_n$

The well-known stable branching rule for $U(n) \downarrow S_n$ is

$$\{\lambda\} \downarrow \langle 1\rangle \otimes \{\lambda/M\}$$

where

$$M = \sigma_1 = \prod_i \frac{1}{1-x_i} \; .$$

To compute the branching coefficients we consider the generating series

$$F_\lambda(z) := \sum_{n\in\mathbf{Z}} \langle 1\rangle \otimes \{(n,\lambda)/M\} z^n := \sum_\mu c_{\lambda\mu}(z)\langle\mu\rangle \; ,$$

and rewrite it as before as

$$F_\lambda(z) = \left[D_{\sigma_1}\sigma_z D_{\lambda_{-1/z}} s_\lambda\right]^\wedge \langle 1\rangle$$

$$= \left[\sigma_z(X+1)s_\lambda\left(X-\frac{1}{z}+1\right)\right]^\wedge \langle 1\rangle$$

$$= (1-z)^{-1}\left[\sigma_z(X)s_\lambda\left(X-\frac{1}{z}+1\right)\right]^\wedge \langle 1\rangle$$

$$= \sigma_1\left(\frac{X}{1-z}\right) \; * \; s_\lambda\left(X-\frac{1}{z}+1\right)^\wedge \langle 1\rangle .$$

Thus, if we define H by

$$\sigma_1 \cdot H = \left[s_\lambda\left(X-\frac{1}{z}+1\right)\right]^\wedge \langle 1\rangle \; ,$$

we arrive at

$$c_{\lambda\mu}(z) = \prod_{k\geq 1}\frac{1}{1-z^k}\; a_{\lambda\mu}(z) \; , \tag{24}$$

where

$$a_{\lambda\mu}(z) = \left\langle s_\mu\left(X+\frac{z}{1-z}\right) \; , \; H\left(\frac{X+1}{1-z}\right)\right\rangle . \tag{25}$$

For $\lambda = (2,1)$, we get

$$\begin{aligned} H\left(\frac{X+1}{1-z}\right) = {} & \frac{(2\,z^5-3\,z^4-2\,z^2+3\,z-1)}{(z^2+z+1)\,(z-1)^3\,z^2}s_0 + \frac{(z^2-3\,z+1)}{(z-1)^3\,z}s_2 \\ & + \frac{(z^2-3\,z+1)}{(z-1)^3\,z}s_{11} + \frac{z}{(z^2+z+1)\,(1-z)^3}s_3 + \frac{(z^2+1)}{(z^2+z+1)\,(1-z)^3}s_{21} \\ & + \frac{z}{(z^2+z+1)\,(1-z)^3}s_{111} + \frac{(3\,z^3-8\,z^2+5\,z-1)}{(z-1)^3\,z^2}s_1 \end{aligned} \tag{26}$$

From this we may compute

$$c_{21,11}(z) = -1 + 3\,z^2 + 15\,z^3 + 42\,z^4 + 102\,z^5 + 215\,z^6 + 425\,z^7 + 785\,z^8 + 1391\,z^9 \\ +2367\,z^{10} + 3912\,z^{11} + 6286\,z^{12} + 9884\,z^{13} + 15221\,z^{14} + O\left(z^{15}\right)$$

$$c_{21,211}(z) = 2\,z^2 + 10\,z^3 + 36\,z^4 + 104\,z^5 + 260\,z^6 + 587\,z^7 + 1229\,z^8 + 2425\,z^9 \\ +4558\,z^{10} + 8231\,z^{11} + 14366\,z^{12} + 24354\,z^{13} + 40247\,z^{14} + O\left(z^{15}\right)$$

Hence, the multiplicity of $(11,2,1,1)$ in the restriction of the irrep $\{12,2,1\}$ of $U(15)$ to S_{15} is equal to 14366, the coefficient of z^{12} in the expansion of $c_{21,211}(z)$.

5. The restriction $O(n) \downarrow S_n$

It is known that the stable branching rule for $O(n) \downarrow S_n$ is given by [11]

$$[\lambda] \downarrow \langle 1 \rangle \otimes \{\lambda/G\}$$

where

$$G = \sigma_1 \cdot \lambda_{-1}[h_2] = M \cdot C = \prod_i \frac{1}{1-x_i} \prod_{i \le j} (1 - x_i x_j)\ .$$

We want to compute the coefficients $c_{\lambda\mu}(z)$ of the generating series

$$F_\lambda(z) := \sum_{n \in \mathbf{Z}} \langle 1 \rangle \otimes \{(n,\lambda)/G\} z^n := \sum_\mu c_{\lambda\mu}(z) \langle \mu \rangle\ .$$

We have

$$\begin{aligned} F_\lambda(z) &= \left[D_{\sigma_1} D_{\lambda_{-1}[h_2]} \sigma_z D_{\lambda_{-1/z}} s_\lambda \right]^\wedge \langle 1 \rangle \\ &= \left[D_{\lambda_{-1}[h_2]} \sigma_z (X+1) s_\lambda \left(X - \frac{1}{z} + 1 \right) \right]^\wedge \langle 1 \rangle \\ &= (1-z)^{-1} \left[D_{\lambda_{-1}[h_2]} \sigma_z (X) s_\lambda \left(X - \frac{1}{z} + 1 \right) \right]^\wedge \langle 1 \rangle\ . \end{aligned}$$

We now use the following properties:

Lemma 5.1. For any symmetric functions f, g,

$$D_{\lambda_{-1}[h_2]}(fg) = \mu \circ \left(D_{\lambda_{-1}[h_2]} \otimes D_{\lambda_{-1}[h_2]} \right) \circ D_{\delta \lambda_{-1}} (f \otimes g)\,,$$

where δ is the comultiplication $\delta(p_k) = p_k \otimes p_k$ (i.e. $\delta(f) = f(XY)$, denoted by Δ^* in [9], p.128) and μ the multiplication operator $\mu(f \otimes g) = fg$.

(Here $\otimes$ means tensor product, not plethysm).

Proof: Let h be an arbitrary symmetric function. Using the duality between multiplication and comultiplication, we have the following sequence of transformations of the scalar product

$$\begin{aligned}
&\left\langle D_{\lambda_{-1}[h_2]}(fg)\,,\, h\right\rangle = \langle f\otimes g\,,\, \Delta(\lambda_{-1}[h_2]\cdot h)\rangle = \langle f\otimes g\,,\, \Delta(\lambda_{-1}[h_2])\Delta(h)\rangle \\
&= \langle f\otimes g\,,\, \lambda_{-1}[h_2\otimes 1 + h_1\otimes h_1 + 1\otimes h_2]\cdot\Delta(h)\rangle \\
&= \left\langle D_{\delta\lambda_{-1}}(f\otimes g)\,,\, \lambda_{-1}[h_2]\otimes\lambda_{-1}[h_2]\cdot\Delta(h)\right\rangle \\
&= \left\langle \mu\circ\left(D_{\lambda_{-1}[h_2]}\otimes D_{\lambda_{-1}[h_2]}\right)\circ D_{\delta\lambda_{-1}}(f\otimes g)\,,\, h\right\rangle\ .
\end{aligned}$$

Lemma 5.2. For any symmetric function f,

$$D_{\delta\lambda_{-1}}\sigma_z\otimes f = \sigma_z\otimes f(X-z)\ .$$

Proof: Taking into account the fact that $D_{s_\nu}h_n = 0$ if $\ell(\nu)>1$ and $D_{s_r}h_n = h_{n-r}$, we have

$$\begin{aligned}
D_{\delta\lambda_{-1}(X)}\sigma_z(X)\otimes f(X) &= \sum_{r\geq 0}(-1)^r D_{s_r}\sigma_z(X)\otimes D_{s_{1^r}}f(X) \\
&= \sigma_z(X)\otimes D_{\lambda_{-z}(X)}f(X) = \sigma_z(X)\otimes f(X-z)\ .
\end{aligned}$$

Also, from the well-known expansion of $\lambda_{-1}[h_2]$ (cf. [9], ex.9, p.78) we get $\lambda_{-1}[h_2] = 1-h_2+$ Schur functions indexed by partitions with more than one part, hence

$$D_{\lambda_{-1}[h_2]}\sigma_z = (1-z^2)\sigma_z\ , \tag{27}$$

which implies that

$$F_\lambda(z) = \frac{(1-z^2)}{(1-z)}\hat{\sigma}_z\langle 1\rangle * U$$

where

$$U := \left[D_{\lambda_{-1}[h_2]}s_\lambda\left(X-\frac{1}{z}+1-z\right)\right]^\wedge\langle 1\rangle = \sigma_1\cdot H\ .$$

Thus, as in the previous section,

$$c_{\lambda\mu}(z) = (1-z^2)\prod_{k\geq 1}\frac{1}{1-z^k}\ a_{\lambda\mu}(z) \tag{28}$$

where

$$a_{\lambda\mu}(z) = \left\langle s_\mu\left(X+\frac{z}{1-z}\right)\ ,\ H\left(\frac{X+1}{1-z}\right)\right\rangle\ . \tag{29}$$

On the example $\lambda = (2,1)$, we get

$$c_{21,11}(z) = -1 + z^2 + 6\,z^3 + 17\,z^4 + 41\,z^5 + 84\,z^6 + 163\,z^7 + 294\,z^8 + 510\,z^9 + 850\,z^{10}$$
$$+1378\,z^{11} + 2172\,z^{12} + 3356\,z^{13} + 5080\,z^{14} + O(z^{15})$$
$$c_{21,2}(z) = 2\,z^2 + 6\,z^3 + 18\,z^4 + 41\,z^5 + 86\,z^6 + 165\,z^7 + 301\,z^8 + 522\,z^9 + 876\,z^{10}$$
$$+1422\,z^{11} + 2253\,z^{12} + 3487\,z^{13} + 5297\,z^{14} + O(z^{15})$$
$$c_{21,221}(z) = z^2 + 3\,z^3 + 12\,z^4 + 36\,z^5 + 95\,z^6 + 221\,z^7 + 478\,z^8 + 966\,z^9 + 1857\,z^{10}$$
$$+3416\,z^{11} + 6065\,z^{12} + 10434\,z^{13} + 17480\,z^{14} + O(z^{15})$$
$$c_{21,5211}(z) = z^7 + 6\,z^8 + 25\,z^9 + 86\,z^{10} + 252\,z^{11} + 663\,z^{12} + 1599\,z^{13} + 3600\,z^{14} + O(z^{15})$$

so that for example, the multiplicity of (95211) in the restriction of the irrep $[13,2,1]$ of $O(18)$ to S_{18} is equal to 1599.

6. The restriction $O(n-1) \downarrow S_n$

The series to be computed here is

$$F_\lambda(z) = \sum_{n\in\mathbf{Z}} \langle 1\rangle \otimes \{(n,\lambda)/C\} = \left[D_{\lambda_{-1}[h_2]}\sigma_z D_{\lambda_{-1/z}} s_\lambda\right]^\wedge \langle 1\rangle \tag{30}$$

A calculation similar to the one of the preceding section shows that

$$F_\lambda(z) = (1-z^2)\left[\sigma_z \cdot D_{\lambda_{-1}[h_2]} s_\lambda\left(X - \frac{1}{z} - z\right)\right]^\wedge \langle 1\rangle \tag{31}$$

and writing as above

$$U = \left[D_{\lambda_{-1}[h_2]} \cdot s_\lambda\left(X - \frac{1}{z} - z\right)\right]^\wedge \langle 1\rangle$$

in the form $U = \sigma_1 \cdot H$ we have

$$c_{\lambda\mu}(z) = (1-z)(1-z^2)\prod_{k\ge 1}\frac{1}{1-z^k} a_{\lambda\mu}(z) \tag{32}$$

where $a_{\lambda\mu}(z)$ is once again given by

$$a_{\lambda\mu}(z) = \left\langle s_\mu\left(X + \frac{z}{1-z}\right)\ ,\ H\left(\frac{X+1}{1-z}\right)\right\rangle\ . \tag{33}$$

For example, with $\lambda = (2,1)$

$$c_{21,11}(z) = z^3 + 3\,z^4 + 7\,z^5 + 15\,z^6 + 29\,z^7 + 52\,z^8 + 89\,z^9 + 147\,z^{10} + 235\,z^{11}$$
$$+366\,z^{12} + 558\,z^{13} + 834\,z^{14} + O(z^{15})$$

$$c_{21,211}(z) = z^2 + 4\,z^3 + 11\,z^4 + 26\,z^5 + 56\,z^6 + 111\,z^7 + 208\,z^8 + 372\,z^9 + 641\,z^{10}$$
$$+1070\,z^{11} + 1739\,z^{12} + 2760\,z^{13} + 4293\,z^{14} + O(z^{15})$$
$$c_{21,421}(z) = z^4 + 3\,z^5 + 12\,z^6 + 37\,z^7 + 98\,z^8 + 231\,z^9 + 507\,z^{10} + 1038\,z^{11}$$
$$+2022\,z^{12} + 3770\,z^{13} + 6781\,z^{14} + O(z^{15})$$

so that, for example, the multiplicity of (5211) in the restriction of the irrep [521] of $O(8)$ to $S(9)$ is equal to 26.

7. A stability property of the coefficients $c_{\lambda\mu}(z)$

The multiplicities of irreps exhibit a certain stability property, if grouped together in a certain natural way. Let us start by looking at some concrete examples for inner plethysm. The remarks are directly applicable to the restrictions (1)-(3), as the previous sections have shown.

The first three examples in table 1 suggest that the sequence of multiplicities becomes constant when increasing the first row of λ and the first row of μ. For example, one has $\langle 1\rangle \otimes \langle n21\rangle \supset 106\langle n-3\rangle$ for $n \geq 10$. We observe a similar stability when increasing the first column, but then have to make shifts on the left hand side by $1, 2, 3, 4, \ldots$ as shown in the fourth example.

To state the property in a precise way, suppose

$$\langle 1\rangle \otimes \{m, \lambda\} \supset C_p^m(\lambda)\langle p, \nu\rangle \text{ and } \langle 1\rangle \otimes \{m+q, \lambda\} \supset C_{p+q}^{m+q}(\lambda)\langle p+q, \nu\rangle. \qquad (34)$$

Then, the $(C_{p+q}^{m+q}(\lambda))_q$ form a finite sequence of integers such that $C_{p+q}^{m+q}(\lambda) = C_{p+q_s}^{m+q_s}(\lambda)$ for all $q \geq q_s$ (and q_s depends on λ and ν only).

This property is shared by $\langle 1\rangle \otimes \{m, \lambda/M\}$, $\langle 1\rangle \otimes \{m, \lambda/G\}$ and $\langle 1\rangle \otimes \{m, \lambda/C\}$.

To see this fix λ and μ and set $\nu := (\mu_2, \mu_3, \ldots)$. Then it is immediate from the definitions that

$$c_{\lambda\mu}(z) = \sum_z C_{\mu_1}^m(\lambda) z^m. \qquad (35)$$

Hence we only need to show a stability property of the $c_{\lambda\mu}(z)$ (or, equivalently, of the $a_{\lambda\mu}(z)$), when μ_1 increases. To get control on the corresponding shifts we rather consider $z^{-\mu_1} c_{\lambda\mu}(z)$ and will show that this expression converges as a formal Laurent series when μ_1 goes to infinity. This property is true for all the restrictions considered before, and it can be proved by using the vertex operator method.

Let u be another indeterminate, commuting with z. Then

$$\begin{aligned}\Gamma_u s_\nu\left(X + \frac{z}{1-z}\right) &= \sigma_u\left(X + \frac{z}{1-z}\right) \cdot s_\nu\left(X - \frac{1}{u} + \frac{z}{1-z}\right)\\ &= \sigma_u\left(\frac{z}{1-z}\right) \cdot \sigma_u\left(X\right) \cdot s_\nu\left(X - \frac{1}{u} + \frac{z}{1-z}\right), \end{aligned} \qquad (36)$$

Table 1. Some multiplicities of $\langle\mu\rangle$ in the inner plethysm $\langle 1\rangle\otimes\{\lambda\}$.

λ	221	321	421	521	621	...
μ	2	3	4	5	6	...
Multiplicity	1	3	5	5	5	...

λ	521	621	721	821	921	10,2,1	11,2,1	...
μ	2	3	4	5	6	7	8	...
Multiplicity	25	58	85	99	104	106	106	...

λ	221	321	421	521	621	...
μ	21	31	41	51	61	...
Multiplicity	2	5	6	6	6	...

λ	5,2,1	7,2,1	10,2,1	14,2,1	19,2,1	25,2,1	...
μ	1^4	1^5	1^6	1^7	1^8	1^9	...
Multiplicity	11	14	17	18	19	19	...

whence

$$\begin{aligned}\sum_i a_{\lambda(i,\nu)}(z)u^i &= \left\langle \Gamma_u s_\nu\left(X+\frac{z}{1-z}\right), H\left(\frac{X+1}{1-z}\right)\right\rangle \\ &= \sigma_u\left(\frac{z}{1-z}\right)\cdot\left\langle \sigma_u(X)s_\nu\left(X-\frac{1}{u}+\frac{z}{1-z}\right), H\left(\frac{X+1}{1-z}\right)\right\rangle \\ &= \sigma_u\left(\frac{z}{1-z}\right)\cdot\left\langle s_\nu\left(X-\frac{1}{u}+\frac{z}{1-z}\right), H\left(\frac{X+1+u}{1-z}\right)\right\rangle \\ &= \sigma_u\left(\frac{z}{1-z}\right)\cdot P_{\lambda\nu}(z,u), \qquad (37)\end{aligned}$$

where

$$P_{\lambda\nu}(z,u) := \sum_{i=b}^{t} r_i(z)u^i := \left\langle s_\nu\left(X - \frac{1}{u} + \frac{z}{1-z}\right)\ ,\ H\left(\frac{X+1+u}{1-z}\right)\right\rangle \quad (38)$$

is a Laurent polynomial in u with rational functions in z as coefficients.
Recall (from [9], ex.5 p.27, e.g.) that

$$\sigma_u\left(\frac{z}{1-z}\right) = \sum_i \frac{z^i}{\prod_{j=1}^{i}(1-z^j)} u^i.$$

Therefore, to get $a_{\lambda\,(\mu_1,\nu)}$, $\mu_1 \geq t$, i.e. the coefficient of u^{μ_1} in

$$\sigma_u\left(\frac{z}{1-z}\right) \cdot P_{\lambda\nu}(z,u),$$

it suffices to consider the coefficient of u^{μ_1} in

$$\left(\sum_{l=\mu_1-t}^{\mu_1-b} \frac{z^l}{\prod_{j=1}^{l}(1-z^j)} u^l\right) \cdot P_{\lambda\nu}(z,u).$$

On the other hand, as a Laurent series in z, this coefficient converges to the coefficient of u^{μ_1} in

$$\frac{z^{\mu_1}}{\prod_{j=1}^{\mu_1}(1-z^j)} u^{\mu_1} \cdot \left(\sum_{l=-t}^{-b} t^l u^l\right) \cdot P_{\lambda\nu}(z,u),$$

when μ_1 goes to infinity. Hence $z^{-\mu_1} c_{\lambda,\mu}(z)$ converges to the constant term (w.r.t. u) in

$$\prod_{j\geq 2}(1-z^j)^{-1} \cdot \prod_{j\geq 1}(1-z^j)^{-1} \cdot \left(\sum_{l=-t}^{-b} t^l u^l\right) \cdot P_{\lambda\nu}(z,u). \quad (39)$$

If we choose $\lambda = (2,1)$ and $\nu := ()$, the empty partition, then

$$P_{\lambda\nu}(z,u) = \frac{z}{(1-z^3)(z-1)^2}\, u^3 + \frac{2\,z-1}{z(1-z)^3}\, u^2 + \frac{1-3z+2z^2+z^3}{z^2(1-z)^3}\, u + \frac{z^4}{(1-z^3)(z-1)^2},$$

and the constant term of

$$z^{-3}u^{-3}\left(1+uz+u^2z^2+u^3z^3\right) P_{\lambda,\nu}(z,u)$$

equals

$$z^{-3}\left(\frac{z}{(1-z^3)(z-1)^2} + \frac{2\,z-1}{(1-z)^3} + \frac{1-3z+2z^2+z^3}{(1-z)^3} + \frac{z^4}{(1-z^3)(z-1)^2}\right)$$
$$= \frac{z^{-1}+2+3z+z^2+z^3}{(1-z)^2(1-z^3)}.$$

Finally for μ_1 large enough the expansion of $z^{-\mu_1} c_{\lambda\mu}(z)$ up to order 7 is

$$z^{-1} + 5 + 17z + 45z^2 + 106z^3 + 230z^4 + 467z^5 + 901z^6 + O(z^7)$$

(the Laurent expansion always starts with $z^{-\ell(\mu)}$). This can be interpreted as follows. The coefficient of z^q in the above expansion is the "stable multiplicity" of $\langle\mu_1\rangle$ in $\langle 1\rangle \otimes \{\mu_1 + q, 2, 1\}$. The coefficient of z^0 is 5 in accordance with the first example of table 1; 106, the coefficient of z^3, appears as the limit in the second example of table 1.

In the fourth example of table 1 the first column of μ increases. We therefore set $\nu := (\mu_2', \mu_3', \ldots)'$, i.e. μ without its first column. As the shifts proceed by steps of $1, 2, 3, \ldots$, we will have to consider the limit $z^{-\binom{\mu_1'-t+1}{2}} c_{\lambda\mu}(z)$, when μ_1' tends to infinity. Here t is a non-negative integer depending on λ and ν only (cf. equation (41) below). Then everything can be proven in the same way as before by using the dual notions.

The dual version of the vertex operator is

$$\sum_i (-1)^{i+1+|\nu|} s_{(i,\nu')'} u^i = \lambda_{-u}(X) s_\nu \left(X + \frac{1}{u}\right). \tag{40}$$

Setting

$$Q_{\lambda\nu}(z,u) := \sum_{i=b}^{t} \tilde{r}_i(z) u^i := \left\langle s_\nu \left(X + \frac{1}{u} + \frac{z}{1-z}\right) \ , \ H\left(\frac{X+1-u}{1-z}\right)\right\rangle \tag{41}$$

and recalling (e.g., from [9], ex.5 p.27) that

$$\lambda_{-u}\left(\frac{z}{1-z}\right) = \sum_i \frac{z^{\binom{i+1}{2}}}{\prod_{j=1}^{i}(1-z^j)} (-u)^i,$$

for $i \to \infty$ only one term, namely

$$\frac{z^{\binom{i-t+1}{2}}}{\prod_{j=1}^{i-t}(1-z^j)} (-u)^{i-t} \cdot \tilde{r}_t(z), \tag{42}$$

dominates and the desired limit $z^{-\binom{\mu_1'-t+1}{2}} c_{\lambda\mu(z)}$ is

$$\prod_{j\geq 2}(1-z^j)^{-1} \cdot \prod_{j \geq 1}(1-z^j)^{-1} \cdot \tilde{r}_t(z). \tag{43}$$

For example, with $\lambda := (2,1)$ and $\nu := ()$, we get $t := 3$,

$$\tilde{r}_t(z) := \frac{z}{(1-z^3)(z-1)^2},$$

and the desired expansion is

$$z + 3z^2 + 8z^3 + 19z^4 + 41z^5 + 82z^6 + 158z^7 + 290z^8 + 516z^9 + O(z^{10}).$$

We can interpret this as follows. The coefficient of z^q gives the stable multiplicity of $\langle 1^{\mu_1'}\rangle$ in $\langle 1\rangle \otimes \{\binom{\mu_1'-2}{2} + q, 2, 1\}$ (recall that $t = 3$). The coefficient of z^4 is 19 in accordance with table 1 above.

8. Concluding remarks

The computation of branching coefficients of $U(n) \downarrow S_n$, $O(n) \downarrow S_n$ and $O(n-1) \downarrow S_n$ plays a key role in symplectic models of nuclei which is a combinatorially explosive problem. Previous algorithms become computationally impossible when the number of nucleons becomes large. The algorithms outlined in this paper overcome the limitations of earlier algorithms and require no use of modification rules. They have the considerable advantage over other methods of being able to yield specific coefficients rather than the complete set of coefficients, most of which, in practical calculations, are redundant.
The stability properties of the branching coefficients and inner plethysm multiplicities find a natural explanation in terms of generating functions.

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