Symmetric Functions and the Symmetric Group 7

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> And yet the mystery of mysteries is to view machines making machines; a spectacle that fills the mind with curious, and even awful, speculation.
> Benjamin Disraeli: Coningsby (1844)

■ 7.1 Group-subgroup decompositions

NB. Herein we follow closely R C King, *Branching rules for classical Lie groups using tensor and spinor methods*, J Phys A:Math.Gen. **8**,429 (1975). Branching rules play an important role in applications of group theory to problems in physics.Consider a group G with elements $\{g,\ldots\}$ and irreducible representations $\{\lambda_{\rm G},\ldots\}$ and a subgroup H i.e. G $\supset$ H with elements $\{h,\ldots\}$ and irreducible representations $\{\mu_{\rm H},\ldots\}$. The restriction of the set of matrices $\{\lambda_{\rm G}(g)\}$ forming the representation $(\lambda)_{\rm G}$ of G to the set$\{\lambda_{\rm G}(h)\}$ yields a representation of H which is generally reducible. If

$$\lambda_{\rm G}(h) = \sum_{\mu_{\rm H}} m_{\lambda_{\rm G}}^{\mu_{\rm H}} \mu_{\rm H}(h) \quad \text{for all} \quad h \in {\rm H} \tag{7.1}$$

then under G $\downarrow$ H we have the *branching rule*

$$ {\rm G} \downarrow {\rm H} \qquad (\lambda)_{\rm G} \downarrow \sum_{\mu_{\rm H}} m_{\lambda_{\rm G}}^{\mu_{\rm H}} (\mu)_{\rm H} \tag{7.2}$$

where the $m_{\lambda_{\rm G}}^{\mu_{\rm H}}$ are known as the *branching rule multiplicities.*

■ 7.2 The Unitary,$U(n)$, and Special Unitary, $SU(n)$, groups

We have already noted the relationship between $S-$functions and the characters of the unitary group and the fact that the irreducible representations of $U(n)$ may be labelled by ordered partitions of integers, thus $\{\lambda\} \equiv \{\lambda_1,\ldots,\lambda_n\}$. (NB in addition to these *covariant* irreducible representations there are irreducible representations involving both covariant and contravariant indices - see the above reference for more details). In practice zero parts are omitted. Under the restriction $U(n) \downarrow SU(n)$ an irreducible representation $\{\lambda\}$ of $U(n)$ remains irreducible and hence we may still label the irreducible representations of $SU(n)$ by ordered partitions with the proviso that irreducible representations of $U(n)$ involving partitions with n positive integers are equivalent to an irreducible representation of $SU(n)$ involving fewer than n positive integers. The equivalence is such that

$$\{\lambda_1,\ldots,\lambda_n\} \equiv \{\lambda_1-\lambda_n,\ldots,0\} \quad if\ \lambda_n > 0 \tag{7.3}$$

Thus for $SU(n)$ the irreducible representations involve at most $n-1$ positive integers. Thus under $U(3) \downarrow SU(3)$ we have, for example, the equivalences

$$\{321\} \equiv \{21\}, \quad \{432\} \equiv \{21\}, \quad \{1^3\} \equiv \{0\}$$

NB. irreducible representations that are inequivalent in $U(n)$ may, under the restriction $U(n) \downarrow SU(n)$ be equivalent to the *same* $SU(n)$ irreducible representation. Irreducible representations of $SU(n)$ that involve partitions $\{\lambda\}$ such that

$$\{\lambda_1,\lambda_2,\ldots,\lambda_n\} \equiv \{\lambda_1-\lambda_n,\lambda_2-\lambda_{n-1},\ldots,0\} \tag{7.4}$$

are said to be *contragredient* to one another and are of the same dimension. Thus in $SU(3)$ we have the pairs $\{2\},\{2^2\}$, $\{31\},\{32\}$ etc. Such pairs are sometimes labelled as $\{\lambda\},\{\bar\lambda\}$. If $\{\lambda\} \equiv \{\bar\lambda\}$ then $\{\lambda\}$ is said to be *self-contragredient*. Thus in $SU(3)$ $\{21\}$, $\{42\}$ are examples of self-contragredient irreducible representations.

■ 7.3 Kronecker products in $U(n)$ and $SU(n)$

The Kronecker product of a pair of irreducible representations of $U(n)$, say $\{\lambda\}$ and $\{\mu\}$ may be resolved into a sum of irreducible representations of $U(n)$ by use of the Littlewood-Richardson rule for $S-$functions to give

$$\{\lambda\} \times \{\mu\} = \sum_{\nu} c^{\nu}_{\lambda\mu}\{\nu\} \tag{7.5}$$

with the proviso that all $\{nu\}$ involving more than n non-zero parts are to be discarded. For example, in $U(3)$ we have upon application of the Littlewood-Richardson rule

$\{21\} \times \{31\} =$

$$\begin{array}{lllll} \{321^2\} & + \{32^2\} & + \{3^21\} & + \{41^3\} & + 2\{421\} \\ + \{43\} & + \{51^2\} & + \{52\} & & \end{array} \tag{7.6}$$

However, the four part partitions $\{321^2\} + \{41^3\}$ must be discarded as null in $U(3)$. NB. for $n \geq 4$ holds as it stands. For the group $SU(3)$ (7.3) must be applied to the three part partitions occurring in (7.6) to give for $SU(3)$ $\{21\} \times \{31\} =$

$$\begin{array}{lllll} \{1\} & + \{2^2\} & + 2\{31\} & + \{4\} & + \{43\} \\ + \{52\} & & & & \end{array} \tag{7.7}$$

■ Exercises

1. Verify that the product of the dimensions on the lhs of (7.7) is equal to the sum of the dimensions on the rhs.
2. Show that in $SU(3)$
$$\{1\} \times \{\bar{1}\} = \{0\} + \{21\}$$
3. Show that in $SU(3)$
$$\{1\} \times \{1\} \times \{1\} = \{0\} + 2\{21\} + \{3\}$$

■ 7.4 The labelling of irreducible representations for the classical Lie groups

Hereon we follow

1. G R E Black, R C King and B G Wybourne, *Kronecker products for compact semisimple Lie groups*, J Phys A:Math.Gen. **16**, 1555 (1983). (BKW)

The partition (λ) of weight w_λ serves to label an irreducible representation $\{\lambda\}$ of $U(n)$ and to specify the symmetry properties of the corresponding w_λth-rank covariant tensor forming a basis for the irreducible representation. This same covariant tensor forms a basis for representations of the subgroups of $U(n)$, including the orthogonal group, $O(n)$, and if n is *even*, the symplectic group $Sp(n)$. In general, these representations are reducible and will be labelled by the partitions $[\lambda]$ and $\langle\lambda\rangle$ respectively.

As well as the tensor irreducible representations labelled by $[\lambda]$, $O(n)$ also has double-valued or spinor representations denoted by $[\Delta; \lambda]$ where Δ is the fundamental spin representation of dimension $2^{[n/2]}$.

For all linear groups there exists amongst the irreducible representations a one-dimensional irreducible representation, denoted by ε, which maps each group element to the value of its determinant. By definition all the elements of $SU(n)$, $SO(n)$ and $Sp(n)$ have determinant $+1$, so that the irreducible representations ε coincide with the identity irreducible representations $\{0\}$, $[0]$ and $\langle 0\rangle$ respectively. However, for $U(n)$ and $O(n)$ this is not the case. For $U(n)$ ε is the irreducible representation $\{1^n\}$ with an inverse

$$\varepsilon^{-1} = \bar{\varepsilon} = \{\bar{1^n}\} \qquad (for\ U(n)) \tag{7.8}$$

For $O(n)$, all the group elements are of determinant ± 1 and hence

$$\varepsilon^{-1} = \varepsilon \qquad and \qquad \varepsilon \times \varepsilon \quad (for\ O(n)) \tag{7.9}$$

The product of ε with any irreducible representation is also an irreducible representation, and inequivalent irreps related by some power of ε are said to be *associated.* For $U(n)$ there are an infinite number of inequivalent associated irreducible representations associated with a given irreducible representation, one of which will be specified by a partition into less than n parts. For instance $\ldots$ $\overline{\{6^2521\}}$, $\overline{\{5^241\}}$, $\overline{\{4^23;1\}} \ldots \{541\}$, $\{6521^2\}$ are all associated irreducible representations of $U(5)$.

Since under $U(n) \downarrow SU(n)$ $\varepsilon \downarrow \{0\}$ it follows that all mutually associated irreducible representations of $U(n)$ give equivalent irreducible representations of $SU(n)$.

In the case of $O(n)$ any given irreducible representation can possess at most one inequivalent associated irreducible representation. Irreducible representations for which the character is zero for all group elements of determinant -1 possess an associate that is equivalent to itself. Such irreducible representations are said to be *self-associate.* For $O(2k)$ all the spinor irreducible representations and all the tensor irreducible representations labelled by exactly k parts are self-associate. The remaining tensor irreducible representations of $O(2k)$ and all irreducible representations, tensor and spinor, of $O(2k+1)$ are not self-associate. Associated pairs of irreducible representations are denoted by $[\lambda]$ and $[\lambda]^*$ and $[\Delta;\lambda]$ and $[\Delta;\lambda]^*$ where

$$[\lambda]^* = \varepsilon \times [\lambda] \qquad \textit{and} \qquad [\Delta;\lambda]^* = \varepsilon \times [\Delta;\lambda] \tag{7.10}$$

Under $O(n) \downarrow SO(n)$ the distinction between an irreducible representation and its associate is lost. However, only those irreducible representations of $O(n)$ which are *NOT* self-associated remain irreducible under $O(n) \downarrow SO(n)$. Each self-associate irreducible representation of $O(2k)$ yields on restriction to $SO(2k)$ two inequivalent irreducible representations of the same dimension which we shall label as $[\lambda]_\pm$ and $[\Delta;\lambda]_\pm$ where in the former case λ is necessarily a partition into k non-zero parts.

Table 7.1 Standard labels for the irreducible representations of the classical groups of rank k.

Group G	Label λ_{G}	Constraint
$U(n)$	$\{\lambda\}$	$\ell_\lambda \le n$
$SU(n)$	$\{\lambda\}$	$\ell_\lambda \le n-1$
$O(2k+1)$	$[\lambda], [\lambda]^*$	$\ell_\lambda \le k$
	$[\Delta;\lambda], [\Delta;\lambda]^*$	$\ell_\lambda \le k$
$SO(2k+1)$	$[\lambda]$	$\ell_\lambda \le k$
	$[\Delta;\lambda]$	$\ell_\lambda \le k$
$O(2k)$	$[\lambda], [\lambda]^*$	$\ell_\lambda < k$
	$[\lambda]$	$\ell_\lambda = k$
	$[\Delta;\lambda]$	$\ell_\lambda \le k$
$SO(2k)$	$[\lambda]$	$\ell_\lambda < k$
	$[\lambda]_+, [\lambda]_-$	$\ell_\lambda = k$
	$[\Delta;\lambda]_+, [\Delta;\lambda]_-$	$\ell_\lambda \le k$
$Sp(2k)$	$\langle\lambda\rangle$	$\ell_\lambda \le k$

■ 7.5 Modification rules

The labels given in Table 7.1 uniquely label the inequivalent irreducible representations of the classical groups. However in many practical applications non-standard labels may arise. In such cases the corresponding character may either vanish or be equal to the character of an irreducible representation specified by a G-standard label or be the negative of such a character.

All the classical groups modification rules can be associated with a common procedure. The key operation is the removal of a continuous boundary strip of boxes of length h from the Young diagram specified by the partition (λ), starting at the foot of the first column and ending in the $c-$th column, to yield symbolically $\lambda - h$. If the resaulting Young diagram is regular then $\lambda - h$ is simply the partition which serves to specify the diagram to which we associate a sign factor $(-1)^c$. If the resulting diagram is not regular then it is discarded since the character vanishes identically. The procedure is repeated until the diagram either corresponds to that of a standard label or vanishes.

Table 7.2. Modification rules for the classical groups. $(p = \ell_\lambda, q = \ell_\mu)$

$U(n), SU(n)$	$\{\bar{\mu};\lambda\} = (-1)^{c+d-1}\{\overline{\mu - h};\lambda - h\}$	$h = p + q - n - 1 \geq 0$
$O(2k+1)$	$[\lambda] = (-1)^{c-1}[\lambda - h]^*$	$h = 2p - 2k - 1 > 0$
	$[\lambda]^* = (-1)^{c-1}[\lambda - h]$	$h = 2p - 2k - 1 > 0$
	$[\Delta;\lambda] = (-1)^{c}[\Delta;\lambda - h]^*$	$h = 2p - 2k - 2 \geq 0$
	$[\Delta;\lambda]^* = (-1)^{c}[\Delta;\lambda - h]$	$h = 2p - 2k - 2 \geq 0$
$SO(2k+1)$	$[\lambda] = (-1)^{c-1}[\lambda - h]$	$h = 2p - 2k - 1 > 0$
	$[\Delta;\lambda] = (-1)^{c}[\Delta;\lambda - h]$	$h = 2p - 2k - 2 \geq 0$
$O(2k)$	$[\lambda] = (-1)^{c-1}[\lambda - h]^*$	$h = 2p - 2k > 0$
	$[\lambda]^* = (-1)^{c-1}[\lambda - h]$	$h = 2p - 2k > 0$
	$[\Delta;\lambda] = (-1)^{c}[\Delta;\lambda - h]$	$h = 2p - 2k - 1 \geq 0$
$SO(2k)$	$[\lambda] = (-1)^{c-1}[\lambda - h]$	$h = 2p - 2k > 0$
	$[\Delta;\lambda] = (-1)^{c}[\Delta;\lambda - h]$	$h = 2p - 2k - 1 \geq 0$
	$[\Box;\lambda] = (-1)^{c-1}[\Box;\lambda - h]$	$h = 2p - 2k - 2 \geq 0$
	$[\Delta;\lambda]_\pm = (-1)^{c}[\Delta;\lambda - h]_\mp$	$h = 2p - 2k - 1 \geq 0$
	$[\Box;\lambda]_\pm = (-1)^{c-1}[\Box;\lambda - h]_{-\mp}$	$h = 2p - 2k - 2 \geq 0$
$Sp(2k)$	$\langle\lambda\rangle = (-1)^{c}\langle\lambda - h\rangle$	$h = 2p - 2k - 2 \geq 0$

■ Exercises

7.1 Verify for $O(6)$ that

$$[3211] = [32]^*, \quad \mathit{and} \quad [\Delta;3211] = -[\Delta;321]$$

7.2 Verify for $Sp(4)$ that

$$\langle 2^2 1^2\rangle = -\langle 2^2\rangle, \quad \mathit{and} \quad \langle 1^6\rangle = -\langle 0\rangle$$

7.3 Verify for $SO(8)$ that

$$[32^2 1^2] = [32^2], \quad [32^2 1^3] = 0, \quad [32^2 1^4] = -[32]$$

■ 7.6 Note on Mixed tensor irreducible representations of $U(n)$

So far we have only discussed the covariant tensor irreducible representations of $U(n)$ which herein will be our principal concern. In addition to the covariant tensor irreducible representations $\{\lambda\}$ there are inequivalent irreducible representations associated with $m-$th rank contravariant tensors specified by $\{\bar{\mu}\}$ and more generally irreducible representations associated with mixed tensors specified by $\{\bar{\mu};\lambda\}$. The technical details are given in BKW.