Symmetric Functions and the Symmetric Group 4

B. G. Wybourne

'Fred!' cried Mr Swiveller, tapping his nose, 'a word to the wise is sufficient for them - we may be good and happy without riches, Fred.'
Charles Dickens *Old Curiosity Shop* (1841).

■ 4.1 The Littlewood-Richardson rule

The product of two $S-$functions can be written as a sum of $S-$functions, viz.

$$s_\mu . s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda \tag{4.1}$$

The Littlewood-Richardson coefficients $c^\lambda_{\mu\nu}$ in Eqs. (3.44) for skew $S-$function and (4.1) are identical, though the summations are of course different. In both cases $|\mu| + |\nu| = |\lambda|$. A rule for evaluating the coefficients $c^\lambda_{\mu\nu}$ was given by Littlewood and Richardson in 1934 and has played a major role in all subsequent developments. The rule may be stated in various ways. We shall state it first in terms of semistandard tableaux and then also give the rule for evaluating the product given in Eq.(4.1) which is commonly referred to as the *outer multiplication* of $S-$functions. In each statement the concepts of a *row-word* and of a *lattice permutation* is used.

■ 4.2 Definition 1 A word

Let T be a tableau. From T we derive a row-word or sequence $w(T)$ by reading the symbols in T from right to left (i.e. as in Arabic or Hebrew) in successive rows starting at the top row and proceeding to the bottom row

Thus for the tableau

$$\begin{array}{|c|c|c|c|c|}
\hline
1 & 1 & 2 & 2 & 3 \\
\hline
2 & 2 & 3 & 3 \\
\cline{1-4}
4 & 4 \\
\cline{1-2}
5 & 6 \\
\cline{1-2}
7 \\
\cline{1-1}
8 \\
\cline{1-1}
\end{array}$$

we have the word $w(T) = 322113322446578$ and for the skew tableau

$$\begin{array}{ccccc}
\cline{3-5}
 & & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{1} \\
\cline{2-5}
 & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{2} & \\
\cline{1-4}
\multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{2} & & & \\
\cline{1-2}
\end{array}$$

we have the word $w(T) = 11122121$.

■ 4.3 Definition 2 A lattice permutation

A word $w = a_1 a_2 \ldots a_N$ in the symbols $1, 2, \ldots, n$ is said to be a lattice permutation if for $1 \leq r \leq N$ and $1 \leq i \leq n-1$, the number of occurrences of the symbol i in $a_1 a_2 \ldots a_r$ is not less than the number of occurrences of $i+1$.

Thus the word $w(T) = 322113322446578$ is clearly not a lattice permutation whereas the word $w(T) = 11122121$ is a lattice permutation. The word $w(T) = 12122111$ is not a lattice permutation since the sub-word 12122 has more twos than ones.

■ Theorem 1 *The value of the coefficient $c^\lambda_{\mu\nu}$ is equal to the number of semistandard tableaux T of shape $F^{\lambda/\mu}$ and content ν such that $w(T)$ is a lattice permutation.*

By *content* ν we mean that each tableau T contains ν_1 ones, ν_2 twos, etc.

■ Example

Let us evaluate the coefficient $c^{\{542\}}_{\{431\}\{21\}}$. We first draw the frame $F^{\{542/21\}}$.

$$\begin{ytableau} \none & \none & ~ & ~ & ~ \\ \none & ~ & ~ & ~ \\ ~ & ~ \end{ytableau}$$

Into this frame we must inject the content of $\{431\}$ i.e. 4 ones, 3 twos and 1 three in such a way that we have a lattice permutation. We find two such numberings

$$\begin{ytableau} \none & \none & 1 & 1 & 1 \\ \none & 1 & 2 & 2 \\ 2 & 3 \end{ytableau} \qquad \begin{ytableau} \none & \none & 1 & 1 & 1 \\ \none & 2 & 2 & 2 \\ 1 & 3 \end{ytableau}$$

and hence $c^{\{542\}}_{\{431\}\{21\}} = 2$. Note that in the evaluation we had a choice, we could have, and indeed more simply, evaluated $c^{\{542\}}_{\{21\}\{431\}}$. In that case we would have drawn the frame $F^{\{542/431\}}$ to get

$$\begin{ytableau} \none & \none & \none & \none & ~ \\ \none & \none & \none & ~ \\ \none & ~ \end{ytableau}$$

Note that in this case the three boxes are disjoint. This skew frame is to be numbered with two ones and one 2 leading to the two tableaux

$$\begin{ytableau} \none & \none & \none & \none & 1 \\ \none & \none & \none & 1 \\ \none & 2 \end{ytableau} \qquad \begin{ytableau} \none & \none & \none & \none & 1 \\ \none & \none & \none & 2 \\ \none & 1 \end{ytableau}$$

verifying the previous result. Theorem 1 gives a direct method for evaluating the Littlewood-Richardson coefficients. These coefficients can be used to evaluate both skews and products. It is sometimes useful to state a procedure for directly evaluating products.

■ Theorem 2 *to evaluate the $S-$function product $\{\mu\}.\{\nu\}$*

1. *Draw the frame F^{μ} and place ν_1 ones in the first row, ν_2 twos in the second row etc until the frame is filled with integers.*
2. *Draw the frame F^{ν} and inject positive integers to form a semistandard tableau such that the word formed by reading from right to left starting at the top row of the first frame and moving downwards along successive rows to the bottom row and then continuing through the second frame is a lattice permutation.*
3. *Repeat the above process until no further words can be constructed.*
4. *Each word corresponds to an $S-$function $\{\lambda\}$ where λ_1 is the number of ones, λ_2 the number of twos etc.*

As an example consider the $S-$function product $\{21\} \cdot \{21\}$.

Step 1 gives the tableau

$$\begin{ytableau} 1 & 1 \\ 2 \end{ytableau}$$

Steps 2 and 3 lead to the eight numbered frames

$$\begin{ytableau} 1 & 1 \\ 2 \end{ytableau} \quad \begin{ytableau} 1 & 1 \\ 3 \end{ytableau} \quad \begin{ytableau} 1 & 2 \\ 2 \end{ytableau} \quad \begin{ytableau} 1 & 2 \\ 3 \end{ytableau} \quad \begin{ytableau} 1 & 3 \\ 2 \end{ytableau} \quad \begin{ytableau} 1 & 3 \\ 4 \end{ytableau} \quad \begin{ytableau} 2 & 3 \\ 3 \end{ytableau} \quad \begin{ytableau} 2 & 3 \\ 4 \end{ytableau}$$

Step 4 then lead to the eight words

112112	112113	112212	112213
112312	112314	112323	112324

from which we conclude that

$$\{21\}.\{21\} = \{42\} + \{41^2\} + \{3^2\} + 2\{321\} + \{31^3\} + \{2^3\} + \{2^2 1^2\}$$

I have made only one non-mathematical discovery in my life, the discovery of the exclusion principle; and that was what I was given the Nobel prize for! (Wolfgang Pauli, 1956)

Dear Professor,
I must have a serious word with you today. Are you acquainted with a certain Mr. Schrödinger, who in the year 1922 (Zeits. fur Phys.,12) described a 'bemerkenswerte Eigenschaft der Quantebahnen'? Are you acquainted with this man? What! You affirm that you know him very well, that you were even present when he did this work and that you were his accomplice in it? That is absolutely unheard of. ......
With hearty greetings, I am
Yours very faithfully

Fritz London

■ 4.4 Relationship to the unitary group

We have explored various symmetric functions indexed by partitions and defined on sets of variables. The variables can admit many interpretations. In some instances we may choose a set of variables $1, q, q^2, \ldots, q^n$ (cf. Farmer, King and Wybourne, *J. Phys. A: Math. Gen.* **21**, 3979 (1988).) or we could even use a set of matrices. The link between $S-$functions and the character theory of groups is such that, if λ is a partition with $\ell(\lambda) \leq N$ and the eigenvalues of a group element, g, of the unitary group U_N are given by $x_j = \exp(i\phi_j)$ for $j = 1, 2, \ldots, N$ then the $S-$function

$$\begin{aligned} \{\lambda\} &= \{\lambda_1\lambda_2\ldots\lambda_N\} = s_\lambda(x) \\ &= s_\lambda(\exp(i\phi_1)\exp(i\phi_2)\ldots\exp(i\phi_N)) \end{aligned} \tag{4.2}$$

is nothing other than the character of g in the irreducible representation of U_N conventionally designated by $\{\lambda\}$.

The Littlewood-Richardson rule gives the resolution of the Kronecker product $\{\mu\} \times \{\nu\}$ of U_N as

$$\{\mu\} \times \{\nu\} = \sum_{|\lambda|=|\mu|+|\nu|} c^{\{\lambda\}}_{\{\mu\}.\{\nu\}}\{\lambda\} \tag{4.3}$$

where the $c^{\{\lambda\}}_{\{\mu\}.\{\nu\}}$ are the usual Littlewood-Richardson coefficients. Equation (4.3) must be modified for partitions λ involving more than N parts. Here the *modification rule* is very simple. We simply discard all partitions involving more than N parts. We shall return to the unitary groups later.

4.5 Reduced notation for the symmetric group

The irreps of the symmetric group $S(N)$ are uniquely labelled by the partitions $\lambda \vdash N$, there being as many irreps of $S(N)$ as there are partitions of N. Consider the following Kronecker products in $S(N)$

$$\begin{aligned} \{21\} \circ \{21\} &= \{3\} + \{21\} + \{1^3\} \\ \{31\} \circ \{31\} &= \{4\} + \{31\} + \{2^2\} + \{21^2\} \\ \{41\} \circ \{41\} &= \{5\} + \{41\} + \{32\} + \{31^2\} \end{aligned}$$

It is apparent that the result stabilises at $N = 4$ and in general we could write

$$\{N-1, 1\} \circ \{N-1, 1\} = \{N, 0\} + \{N-1, 1\} + \{N-2, 2\} + \{N-2, 1^2\} \tag{4.4}$$

The above result would hold for all N provided we apply the modification rules to any non-standard $S-$functions. Thus

$$\begin{aligned} \{21\} \circ \{21\} &= \{3\} + \{21\} + \{12\} + \{1^3\} \\ &= \{3\} + \{21\} + \{1^3\} \end{aligned}$$

since $\{12\} = -\{12\} = 0$.

Equation (4.4) could be rewritten as

$$\langle 1\rangle \circ \langle 1\rangle = \langle 0\rangle + \langle 1\rangle + \langle 2\rangle + \langle 1^2\rangle \tag{4.5}$$

The above equation is an example of the use of *reduced notation* (cf. Scharf, Thibon and Wybourne, *J. Phys. A: Math. Gen.* **26, 7461 (1993) (STW)**, Butler and King, *J. Math. Phys.* **14, 1176 (1973)(BK)** and references therein.) which makes use of the fact that the symmetric group is a subgroup of the general linear group $Gl(N)$. In the reduced notation the irrep label $\{\lambda\} = \{\lambda_1, \lambda_2, \ldots, \lambda_p\}$ in $S(N)$ is replaced by $\langle\lambda\rangle = \langle\lambda_2, \ldots, \lambda_p\rangle$. Given any irrep $\langle\mu\rangle$ in reduced notation it can be converted back into a standard irrep of $S(N)$ by prefixing it with a part $N - |\mu|$. For example, an irrep $\langle 21\rangle$ in reduced notation corresponds in $S(6)$ t0 $\{321\}$ or $\{921\}$ in $S(12)$. If $N - |\mu| \geq \mu_1$ then the irrep $\{N - |\mu|, \mu\}$ is assuredly a standard irrep of $S(N)$. However, if $N - |\mu|\langle\mu_1$ then the resulting irrep $\{N - |\mu|, \mu\}$ is non-standard and must be converted into standard form.

■ **4.5 Reduced Kronecker products for $S(N)$**

BK have, following Littlewood, given the reduced Kronecker product as

$$\langle\lambda\rangle \circ \langle\mu\rangle = \sum_{\alpha,\beta,\gamma} \langle(\{\lambda\}/\{\alpha\}\{\beta\}) \cdot (\{\mu\}/\{\alpha\}\{\gamma\}) \cdot (\{\beta\} \circ \{\gamma\})\rangle \tag{4.6}$$

where the $\cdot$ signifies ordinary Littlewood-Richardson multiplication of the relevant $S-$function.

■ **4.6 Exercises**

4.1 Show that $\langle 21\rangle \circ \langle 31\rangle$ evaluates as

$$\begin{array}{llllll}
\langle 6\rangle & +\langle 52\rangle & +\langle 51^2\rangle & +4\langle 51\rangle & +3\langle 5\rangle & +\langle 43\rangle \\
+2\langle 421\rangle & +6\langle 42\rangle & +\langle 41^3\rangle & +6\langle 41^2\rangle & +10\langle 41\rangle & +5\langle 4\rangle \\
+\langle 3^21\rangle & +3\langle 3^2\rangle & +\langle 32^2\rangle & +\langle 321^2\rangle & +8\langle 321\rangle & +11\langle 32\rangle \\
+4\langle 31^3\rangle & +12\langle 31^2\rangle & +13\langle 31\rangle & +5\langle 3\rangle & +2\langle 2^3\rangle & +3\langle 2^21^2\rangle \\
+9\langle 2^21\rangle & +8\langle 2^2\rangle & +\langle 21^4\rangle & +6\langle 21^3\rangle & +11\langle 21^2\rangle & +9\langle 21\rangle \\
+3\langle 2\rangle & +\langle 1^5\rangle & +3\langle 1^4\rangle & +4\langle 1^3\rangle & +3\langle 1^2\rangle & +\langle 1\rangle
\end{array}$$

4.2 Use the above result to deduce that for $S(5)$ $\{221\} \circ \{221\}$ evaluates as

$$\begin{array}{llllll}
\{5\} & +\{41\} & +\{32\} & +\{31^2\} & +\{2^21\} & +\{21^3\}
\end{array}$$

4.3 Show that in $S(8)$ $\{521\} \circ \{431\}$ evaluates as

$$\begin{array}{llllll}
\{71\} & +3\{62\} & +3\{61^2\} & +4\{53\} & +9\{521\} & +4\{51^3\} \\
+2\{4^2\} & +9\{431\} & +7\{42^2\} & +10\{421^2\} & +3\{41^4\} & +5\{3^22\} \\
+6\{3^21^2\} & +7\{32^21\} & +5\{321^3\} & +\{31^5\} & +\{2^4\} & +2\{2^31^2\} \\
+\{2^21^4\} & & & & &
\end{array}$$

■ **4.7 Kronecker products for two-row partitions**

In quantum chemistry the Pauli exclusion principle restricts interest to irreps of $S(N)$ indexed by partitions into at most two parts. In terms of reduced notation two-row shapes become one-row shapes via the equivalence

$$\{N-k,k\} \circ \{N-\ell,\ell\} \equiv \langle k\rangle \circ \langle\ell\rangle \tag{4.7}$$

From Eq. (4.7) we are led directly to

$$\begin{aligned}
\langle k\rangle \circ \langle\ell\rangle &= \sum_{p=0}^{min(k,\ell} \sum_{q=0}^{p} \langle\{k-p\} \cdot \{\ell-p\} \cdot \{p-q\}\rangle \\
&= \sum_{\lambda} c^{\nu}_{\lambda\mu} \langle\lambda\rangle
\end{aligned} \tag{4.8}$$

The possible shapes for λ are severely constrained. The number of rows cannot exceed three. The multiplicity to be associated with a given shape λ can be readily determined by drawing the shape and then filling the cells, in accordance with the Littlewood-Richardson rule, with say $k-p$ circles $\circ$, $\ell-p$ stars $\circ$ and $p-q$ diamonds $\diamond$, where

$$k + \ell - p + q = \lambda_1 + \lambda_2, \ldots \tag{4.9}$$

Repeated cells will be marked with dots $\cdot$. Consider the shape characterised by the one-row (m), the only case relevant to quantum chemistry. A typical filling is shown below:

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline \circ & \circ & \cdot & \cdot & \cdot & \circ & \circ & \circ & \cdot & \cdot & \cdot & \circ & \diamond & \diamond & \cdot & \cdot & \cdot & \diamond \\ \hline \end{array}$$

From which we can deduced that $c_{\langle k\rangle\langle \ell\rangle}^{\langle m\rangle}$ is the number of partitions of $k+\ell-m$ into two parts (p,q) with $p \geq q$ and $\ell \geq p$ leading to

$$c_{\langle k\rangle\langle \ell\rangle}^{\langle m\rangle} = \frac{1}{2}(\ell - k + m + 2) \quad \textbf{for} \quad k > m \tag{4.9a}$$

$$c_{\langle k\rangle\langle \ell\rangle}^{\langle m\rangle} = \frac{1}{2}(k + \ell - m + 2) \quad \textbf{for} \quad m \geq k \tag{4.9b}$$

and the coefficient symmetry

$$c_{\langle k\rangle\langle \ell\rangle}^{\langle m\rangle} = c_{\langle k\rangle\langle \ell\rangle}^{\langle 2k-m\rangle} \tag{4.10}$$

Exercises

Show that

$$\begin{aligned}\langle 4\rangle \circ \langle 6\rangle =& \langle 10\rangle + \langle 9\rangle + 2\langle 8\rangle + 2\langle 7\rangle + 3\langle 6\rangle + 2\langle 5\rangle \\ &+ 2\langle 4\rangle + \langle 3\rangle + \langle 2\rangle\end{aligned}$$

and hence for $S(12)$

$$\{84\} \circ \{6^2\} = \{10\ 2\} + \{84\} + \{6^2\}$$

Check that the above result is dimensionally correct.