Symmetric Functions and the Symmetric Group 12
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12.1 Introduction

In this lecture I want to continue to discuss $N-$noninteracting fermions or bosons in a isotropic d-dimensional harmonic oscillator leading up to partition functions for such systems. This is of course just the beginning as one must eventually put in realistic interactions taking the non-interacting case as a basis. We want to pay particular attention to the enumeration of the *complete* set of basis states. This can be done in a number of ways each related to the other by some unitary transformation. One can start by considering the non-compact metapletic group $Mp(2Nd)$ and then working down through various compact and non-compact subgroups as in

1. K Grudziński and B G Wybourne, *Symplectic models of $n-$particle systems*, Rep. Math. Phys. **38**, 251-66.

However, in this lecture I shall try to keep the approach relatively simple, starting with a single fermion or boson in a isotropic d-dimensional harmonic oscillator to establish a basis and then discuss the case of various approaches to the problem of N identical noninteracting bosons or fermions. We will assume that the spin of the single particle is s_b (or s_f) for the boson (or the fermion). I shall assume familiarity with earlier lectures in this series.

12.2 Single-particle states for a isotropic d-dimensional harmonic oscillator

We introduce three schemes for describing a single particle (fermion or boson) in a isotropic d-dimensional harmonic oscillator.

1. A non-compact scheme

The infinite set of spatial states span the basic infinite dimensional unitary harmonic series irreducible representation $\tilde{\Delta}$ and classify the states under the scheme

$$SU(2) \times (Mp(2d) \supset Sp(2d, \Re) \supset U(d) \supset O(d) \supset \dots U(1)) \tag{1}$$

Note that we have a direct product with $SU(2)$ being the group describing the spin part of our wavefunction and the $Mp(2d)$ group and its subgroups the spatial part. Recalling that under $Mp(2d) \supset Sp(2d, \Re)$

$$\tilde{\Delta} \to \Delta_+ + \Delta_- \tag{2}$$

while under $Sp(2d, \Re) \supset U(d)$

$$\Delta_+ \to M_+ \tag{3a}$$

$$\Delta_- \to M_- \tag{3b}$$

with

$$M_+ = \sum_{m=0}^{\infty} \{2m\} \tag{4a}$$

$$M_- = \sum_{m=0}^{\infty} \{2m+1\} \tag{4b}$$

$$M = M_+ + M_- = \sum_{m=0}^{\infty} \{m\} \tag{4c}$$

We also recall that under $U(d) \supset O(d)$ we have the general result

$$\{\lambda\} \to [\lambda/D] \tag{5}$$

where D is the infinite $S-$function series

$$D = \sum_{\delta}^{\infty} \{\delta\} \tag{6}$$

Where the summation is over all partitions (δ) whose parts are all *even*.

For details see

2. K Grudziński and B G Wybourne, Plethysm for the noncompact group Sp(2n,R) and new S-function identities J Phys A:Math.Gen.**29**, 6631-41 (1996).
3. R C King and B G Wybourne, *Holomorphic discrete series* ... J Phys A:Math.Gen.**18**, 3113-39 (1985).
4. R C King and B G Wybourne, *Products and symmetrized powers of irreducible representations of* $Sp(2n,\Re)$ *and their associates* J Phys A:Math.Gen.**31**, 6669-89 (1998).
5. R C King and B G Wybourne, *Analogies between finite-dimensional irreps of* $SO(2n)$ *and infinite-dimensional irreps of* $Sp(2n,\Re)$, J. Math. Phys.**41**, 5002-19 (2000).
6. R C King and B G Wybourne, *Analogies between finite-dimensional irreps of* $SO(2n)$ *and infinite-dimensional irreps of* $Sp(2n,\Re)$*, Part II:Plethysms,* J. Math. Phys. **41**, 5656-90 (2000).

■ 2. The $SU(2)\times U(d)$ scheme

In this scheme the spin s belongs to the group $SU(2)$ and spans the $SU(2)$ irreducible representation $\{2s\}$ while the spatial parts span the infinite set of irreducible representations of $U(d)$ labelled by one-part partitions $\{m\}$ so we can symbolically designate the $SU(2)\times U(d)$ single particle states by

$$\{2s\}\times M=\sum_{m=0}^{\infty}\{2s\}\times\{m\} \tag{7}$$

the distinction between bosons and fermions being made at the $SU(2)$ level. The *even* parity states will be associated with the *even* values of m and the *odd* parity states with the *odd* values of m.

■ 3. The $U(1)\times U(d)$ scheme

In this scheme we work at the spin projection level where the different m_s states span one-dimensional irreducible representations of the Abelian group $U(1)$ which we will choose to label as $\{m_s\}$ and remember that for $U(1)$ the Kronecker products are such that

$$\{p\}\times\{q\}=\{p+q\} \tag{8a}$$

while for symmetrized powers (or plethysms)

$$\{p\}\otimes\{\lambda\}=\begin{cases}0 & \text{if } \ell(\lambda)>1\\ p\times\lambda_1 & \text{if } \ell(\lambda)=1\end{cases} \tag{8b}$$

the complete set of single particle states will span the reducible representation

$$\sum_{m_s=-s}^{m_s=s}\{m_s\}\times M \tag{9}$$

■ 12.3 $N-$noninteracting particles in a isotropic d-dimensional harmonic oscillator

The distinction between bosons and fermions becomes crucial when we consider more than one particle. Throughout we shall assume that the N particles are indistinguishable. The basic ansatz is that for bosons the $N-$particle wavefunctions must be totally *symmetric* with respect to all permutations of the N particles while for fermions the $N-$particle wavefunctions must be totally *antisymmetric* with respect to all permutations of the N particles. In other words boson wavefunctions are permanental while those of fermions are determinantal. If our wavefunction is constructed as products of spin and spatial parts then the symmetrization of the spin and spatial parts need not themselves be symmetric (or antisymmetric) but their product must follow the correct statistics. Before continuing a brief diversion to recall some results involving plethysms, recalling parts of earlier lectures. Those unfamiliar with the properties of plethysms might consult some of the references listed in my publications, particularly publications 32,35,39,45,83,88,154 and references contained therein.

■ Plethysm for direct products of groups

In many applications we are involved with the direct product of two groups (more than two poses no new difficulties) say, $\mathcal{G}\times\mathcal{G}'$ with irreducible representations $A_{\mathcal{G}}\times B_{\mathcal{G}'}$ and we need to determine the

$\mathcal{G}\times\mathcal{G}'$ content of $N-$fold product of an irreducible representation say $(A\times B)^{\times N}$ (henceforth we drop the subscripts) and extract the part of the product symmetrized according to the permutational symmetry $\{\lambda\}$. In terms of plethysm we have

$$(A\times B)\otimes\{\lambda\}=\sum_{\rho}(A\otimes\{\rho\cdot\lambda\})\times(B\otimes\{\rho\}) \tag{10}$$

where $\{\rho\cdot\lambda\}$ signifies a $S-$function inner product which is null unless the partitions (ρ) and (λ) are of the same weight, i.e. $|\rho|=|\lambda|$. Two special cases are of interest

$$\{\rho\cdot\{\lambda\}=\begin{cases}\{\rho\} & \text{if } \{\lambda\}=\{N\} \text{ and } |\rho|=|\lambda| \\ \{\rho'\} & \text{if } \{\lambda\}=\{1^N\} \text{ and } |\rho|=|\lambda|\end{cases} \tag{11}$$

where the partition (ρ') is conjugate to (ρ).

By way of example we have

$$\begin{aligned}(A\times B)\otimes\{4\}=&(A\otimes\{4\})\times(B\otimes\{4\})+(A\otimes\{31\})\times(B\otimes\{31\})+(A\otimes\{2^2\})\times(B\otimes\{2^2\})\\&+(A\otimes\{21^2\})\times(B\otimes\{21^2\})+(A\otimes\{1^4\})\times(B\otimes\{1^4\})\end{aligned} \tag{12a}$$

$$\begin{aligned}(A\times B)\otimes\{1^4\}=&(A\otimes\{4\})\times(B\otimes\{1^4\})+(A\otimes\{31\})\times(B\otimes\{21^2\})+(A\otimes\{2^2\})\times(B\otimes\{2^2\})\\&+(A\otimes\{21^2\})\times(B\otimes\{31\})+(A\otimes\{1^4\})\times(B\otimes\{4\})\end{aligned} \tag{12b}$$

In many cases of interest only some of the terms in the right-hand-side of (12) will be non-null. This is particularly the case when one of the groups is of low rank, e.g, $SU(2)$ or $U(1)$. To be specific, let us henceforth consider bosons of spin $s_b=1$ and fermions of spin $s_f=\frac{1}{2}$. In this case the boson spin spans the $\{2\}$ irreducible representation of $SU(2)$ while the fermion spin spans the $\{1\}$ irreducible representation of $SU(2)$. There is no difficulty in going to higher spin states.

■ **The $SU(2)\times Mp(2d)$ scheme**

In this scheme the single particle spans the representation $\{2s\}\times\tilde{\Delta}$ of $SU(2)\times Mp(2d)$ and for $N-$noninteracting particles we have

$$(\{2\}\times\tilde{\Delta})\otimes\{N\}=\sum_{\rho\vdash N}(\{2\}\otimes\{\rho\})\times(\tilde{\Delta}\otimes\{\rho\}) \quad \text{for bosons} \tag{13a}$$

$$(\{1\}\times\tilde{\Delta})\otimes\{1^N\}=\sum_{\rho\vdash N}(\{1\}\otimes\{\rho'\})\times(\tilde{\Delta}\otimes\{\rho\}) \quad \text{for fermions} \tag{13b}$$

Let us consider the evaluation of the $SU(2)$ plethysms, first for fermions and then for bosons.

We have noted earlier that for fermions of spin $s_f=\frac{1}{2}$ that $\{1\}\otimes\{\rho\}=\{\rho\}$ and that the partition (ρ) can involve at most two parts and in $SU(2)$ we have the irreducible representation equivalence

$$\{\rho_1,\rho_2\}\equiv\{\rho_1-\rho_2\} \tag{14}$$

leading to

$$(\{1\}\times\tilde{\Delta})\otimes\{1^N\}=\sum_{S=S_{min}}^{\frac{N}{2}}{}^{2S+1}(\tilde{\Delta}\otimes\{2^{\frac{N}{2}-S}1^{2S}\}) \tag{15}$$

$$S_{min}=\begin{cases}\frac{1}{2} & \text{if } N \text{ is odd}\\ 0 & \text{if } N \text{ is even}\end{cases} \tag{16}$$

Thus for $N=4$ fermions we have

$$(\{1\}\times\tilde{\Delta})\otimes\{1^4\}={}^5(\tilde{\Delta}\otimes\{1^4\})+{}^3(\tilde{\Delta}\otimes\{21^2\})+{}^1(\tilde{\Delta}\otimes\{2^2\}) \tag{17}$$

Recalling the isomorphisms between $SO(3)$ and its covering group $SU(2)$ we have under $SU(2)\sim SO(3)$ $\{2\}\sim[1]$ leading to

$$\{2\}\otimes\{\rho\}\sim[\rho/D] \tag{18}$$

The right-hand-side of (18) gives the spins for each partition (ρ) appearing in (13a). Furthermore, (ρ) can involve at most three non-zero parts and those involving three non-zero parts are equivalent to a partition with two or less parts via

$$\{\rho_1, \rho_2, \rho_3\} \equiv \{\rho_1 - \rho_3, \rho_2 - \rho_3\} \tag{19}$$

NB If $[\rho/D]$ leads to partitions involving more than one non-zero part then the $SO(3)$ modification rules need to be applied. Assuming (19) has been applied leaving a $SO(3)$ non-standard irreducible representation $[a, b]$ then

$$[a, b] \equiv \begin{cases} 0 & \text{if } b \geq 2 \\ \\ [a] & \text{if } b = 1 \end{cases} \tag{20}$$

with the above in mind we can use (13a) to give for four spin 1 bosons

$$(\{2\} \times \tilde{\Delta}) \otimes \{4\} =^{(9+5+1)} (\tilde{\Delta} \otimes \{4\}) +^{(7+5+3)} (\tilde{\Delta} \otimes \{31\}) +^{3} (\tilde{\Delta} \otimes \{21^2\}) +^{(5+1)} (\tilde{\Delta} \otimes \{2^2\}) \tag{21}$$

Where again the multiplicities $(2S + 1)$ are given as left superscripts. To complete the examples of this scheme one should evaluate the various plethysms for the relevant metapletic group and then branch through the various subgroups. We shall not do that at this time.

- **The $SU(2) \times U(d)$ scheme**

In this scheme one starts with (7) and evaluates the relevant plethysms as in the previous scheme. For the spin part there are no changes. The $U(d)$ irreducible representations are combined as the single infinite dimensional reducible representation M. Thus for N spin $\frac{1}{2}$ fermions we have from noting (15)

$$(\{1\} \times M) \otimes \{1^N\} = \sum_{S=S_{min}}^{\frac{N}{2}} {}^{2S+1}(M \otimes \{2^{\frac{N}{2}-S} 1^{2S}\}) \tag{22}$$

and for four fermions

$$(\{1\} \times M) \otimes \{1^4\} =^{5} (M \otimes \{1^4\}) +^{3} (M \otimes \{21^2\}) +^{1} (M \otimes \{2^2\}) \tag{23}$$

as indeed found and expanded in the previous lecture. For N bosons of spin 1 the result comes from (21) by simply replacing $\tilde{\Delta}$ by M throughout.

- **The $U(1) \times U(d)$ scheme**

In this scheme we treat spin at the level of its projection m_s. Clearly in each scheme there must be a complete accounting of all the quantum states and respecting symmetrization. In the case of fermions of spin $\frac{1}{2}$ we have for N particles

$$\big((\{\tfrac{1}{2}\} \times M) + (\{-\tfrac{1}{2}\} \times M)\big) \otimes \{1^N\} = \sum_{x=0}^{N} \big((\{\tfrac{1}{2}\} \times M) \otimes \{1^{N-x}\}\big) \times \big((\{-\tfrac{1}{2}\} \times M) \otimes \{1^x\}\big) \tag{24}$$

Noting (8a) and (8b), we can rewrite (24) as

$$\big((\{\tfrac{1}{2}\} \times M) + (\{-\tfrac{1}{2}\} \times M)\big) \otimes \{1^N\} = \sum_{x=0}^{N} \big(\{\tfrac{N-x}{2}\} \times (M \otimes \{1^{N-x}\})\big) \times \big(\{-\tfrac{x}{2}\} \times (M \otimes \{1^x\})\big) \tag{25}$$

Notice that (25) involves the product of two terms, the first term, $\big(\{\frac{N-x}{2}\} \times (M \otimes \{1^{N-x}\})\big)$, involves states with spin projection $M_S = \frac{N-x}{2}$ (spin-up) which are antisymmetric in their spatial part while the second term, $\big(\{-\frac{x}{2}\} \times (M \otimes \{1^x\})\big)$,involves states with spin projection $M_S = -\frac{x}{2}$ (spin-down) which are again antisymmetric in their spatial part. Equation (25) involves Kronecker products in $U(1)$ and in $U(d)$ and (25) may be rearranged as

$$\big((\{\tfrac{1}{2}\} \times M) + (\{-\tfrac{1}{2}\} \times M)\big) \otimes \{1^N\} = \sum_{x=0}^{N} \big(\{\tfrac{N-x}{2}\} \times \{-\tfrac{x}{2}\}\big) \times \big((M \otimes \{1^{N-x}\}) \times (M \otimes \{1^x\})\big) \tag{26}$$

The first Kronecker product can be evaluated using (8a) to give

$$\big(\{\tfrac{N-x}{2}\} \times \{-\tfrac{x}{2}\}\big) = \{\tfrac{N}{2} - x\} \tag{27}$$

and the second using the plethysm property

$$(A \otimes \{\lambda\}) \times (A \otimes \{\mu\}) = A \otimes (\{\lambda\} \times \{\mu\}) \tag{28}$$

leading to

$$\big((M \otimes \{1^{N-x}\}) \times (M \otimes \{1^x\})\big) = M \otimes (\{1^{N-x}\} \cdot \{1^x\}) \tag{29}$$

with the $\cdot$ implying ordinary $S-$function multiplication. Combining (27) and (29) in (26) finally gives

$$\big((\{\tfrac{1}{2}\} \times M) + (\{-\tfrac{1}{2}\} \times M)\big) \otimes \{1^N\} = \sum_{x=0}^{N} \{\tfrac{N}{2} - x\} \times \big(M \otimes (\{1^{N-x}\} \cdot \{1^x\})\big) \tag{30}$$

For four fermions of spin $\frac{1}{2}$ we obtain

$$\begin{aligned}&\big((\{\tfrac{1}{2}\} \times M) + (\{-\tfrac{1}{2}\} \times M)\big) \otimes \{1^4\}\\ &= \{2\} \times (M \otimes (\{1^4\} \cdot \{0\})) + \{1\} \times (M \otimes (\{1^3\} \cdot \{1\})) + \{0\} \times (M \otimes (\{1^2\} \cdot \{1^2\}))\\ &+ \{-1\} \times (M \otimes (\{1\} \cdot \{1^3\})) + \{-2\} \times (M \otimes (\{0\} \cdot \{1^4\}))\end{aligned} \tag{31a}$$

$$\begin{aligned}&= (\{2\} + \{-2\}) \times (M \otimes \{1^4\}) + (\{1\} + \{-1\}) \times (M \otimes (\{1^3\} \cdot \{1\}))\\ &+ \{0\} \times (M \otimes (\{1^2\} \cdot \{1^2\}))\end{aligned} \tag{31b}$$

$$\begin{aligned}&= (\{2\} + \{-2\}) \times (M \otimes \{1^4\}) + (\{1\} + \{-1\}) \times (M \otimes (\{1^4\} + \{21^2\}))\\ &+ \{0\} \times (M \otimes (\{1^4\} + \{21^2\} + \{2^2\}))\end{aligned} \tag{31c}$$

Comparison with (17) and (23) shows, as should be, that the same number of quantum states are obtained in each scheme. We note that the above scheme was first used by Shudeman[6] to determine the states arising from configurations of equivalent electrons ℓ^N though without using group theory. It was then used by Judd[7] to recast atomic shell theory, Judd giving a group formulation to the scheme and naming it $LL-$coupling. I have given further details[9].

Let us return to the spin 1 bosons. Each boson has three spin states ($M_S = 0,\ \pm 1$) that can be described by the $U(1)$ irreducible representations $\{1\},\ \{0\},\ \{-1\}$. For $N-$noninteracting bosons we have from plethysm

$$\begin{aligned}&(\{1\} \times M + \{0\} \times M + \{-1\} \times M) \otimes \{N\}\\ &= \sum_{x=0}^{N}\sum_{y=0}^{x} [((\{1\} \times M) \otimes \{N-x\}) \times ((\{0\} \times M) \otimes \{x-y\}) \times ((\{-1\} \times M) \otimes \{y\})]\end{aligned} \tag{32a}$$

$$= \sum_{x=0}^{N}\sum_{y=0}^{x} [(\{N-x\} \times (M \otimes \{N-x\})) \times (\{0\} \times (M \otimes \{x-y\})) \times (\{-y\} \times (M \otimes \{y\}))] \tag{32b}$$

$$= \sum_{x=0}^{N}\sum_{y=0}^{x} [\{N-x-y\} \times (M \otimes (\{N-x\} \cdot \{x-y\} \cdot \{y\})] \tag{32c}$$

where in (32c) the spin projection quantum number, M_S is

$$M_S = N - x - y \tag{33}$$

For brevity, let us define

$$M_S^{\uparrow\downarrow}(S) = \begin{cases} \{S\} + \{-S\} & \text{if } S > 0 \\ \{0\} & \text{if } S = 0 \end{cases} \tag{34}$$

For four spin 1 bosons we have from (32c)

$$
\begin{aligned}
&(\{1\}\times M+\{0\}\times M+\{-1\}\times M)\otimes\{4\}\\
&=M_S^{\uparrow\downarrow}(4)(M\otimes\{4\})+M_S^{\uparrow\downarrow}(3)(M\otimes\{3\}\cdot\{1\})+M_S^{\uparrow\downarrow}(2)(M\otimes(\{3\}\cdot\{1\}+\{2\}\cdot\{2\})\\
&+M_S^{\uparrow\downarrow}(1)(M\otimes(\{2\}\cdot\{1\}\cdot\{1\}+\{3\}\cdot\{1\})+M_S^{\uparrow\downarrow}(0)(M\otimes(\{4\}+\{2\}\cdot\{2\}+\{2\}\cdot\{1\}\cdot\{1\})
\end{aligned}
\tag{35a}
$$

$$
\begin{aligned}
&=M_S^{\uparrow\downarrow}(4)(M\otimes\{4\})+M_S^{\uparrow\downarrow}(3)(M\otimes(\{4\}+\{31\}))+M_S^{\uparrow\downarrow}(2)(M\otimes(2\{4\}+2\{31\}+\{2^2\}))\\
&+M_S^{\uparrow\downarrow}(1)(M\otimes(2\{4\}+3\{31\}+\{2^2\}+\{21^2\}))+M_S^{\uparrow\downarrow}(0)(M\otimes(3\{4\}+3\{31\}+2\{2^2\}+\{21^2\}))
\end{aligned}
\tag{35b}
$$

which is consistent with the M_S projection of the spins found in (21).

I am indebted to Jürgen Schnack for pointing out to me the relevance of the scheme for computing partition functions.

7. C L B Shudeman, J. Franklin Inst. **224**, 501 (1937).

8. B R Judd, *Atomic Shell Theory Recast*, Phys. Rev. **162**, 28-37 (1967).

9. B G Wybourne, *Coefficients of fractional parentage and LL-Coupling*, J. de Phys. **30**, 35-8 (1969).

Additional information on boson-fermion relationships, not covered in these lectures, may be found in

10. B G Wybourne, *Hermite's Reciprocity Law and the Angular Momentum States of Equivalent Particle Configurations*, J. Math. Phys. **10**, 467-71 (1969).

11. B G Wybourne, *Statistical and Group Properties of the Fractional Quantum Hall Effect* (SSPCM'2000, 31 August - 6 September 2000, Myczkowce, Poland) Singapore: World Scientific (In Press).