Symmetric Functions and the Symmetric Group 1

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He who can, does; he who cannot teaches.
George Bernard Shaw, *Man & Superman* (1903)
Those who can, do, those who can't, attend conferences.
Daily Telegraph 6th August, (1979)

Introduction

These are rough notes on symmetric functions and the symmetric group and are given purely as a guide. I intend to outline some of the basic properties of symmetric functions as relevant to application to problems in chemistry and physics. The partition of integers plays a key role and we shall first make remarks on partitions in order to establish notation and then go on to consider the standard symmetric functions, their definitions and their generators. This will lead to the important symmetric functions known as $S-$functions so named in honour of Schur. Important properties to be discussed will be their outer and inner multiplication and plethysm. At that stage we can start to look at specific applications.

Partitions

An *ordered* partition λ of *length* $p = \ell(\lambda)$, corresponds to an ordered set of p integers

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p) \tag{1}$$

such that

$$\lambda_1 \geq \lambda_2, \ldots \geq \lambda_p \geq 0 \tag{2}$$

Unless otherwise stated we shall mean by a partition an ordered partition. Normally we shall omit trailing zeros.

The *weight* ω_λ of a partition λ will be defined as the sum of its parts.

$$\omega_\lambda = |\, l| = \lambda_1 + \lambda_2, + \ldots + \lambda_p \tag{3}$$

If $|\, l| = n$ then λ is said to be a partition of n. We shall denot the set of partitions $\lambda \vdash n$ as $\mathcal{P}_n$ and the set of all partitions by $\mathcal{P}$. Thus

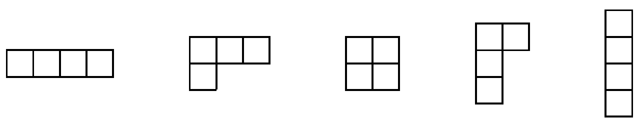
$$\mathcal{P}_4 \supseteq \{(4), (31), (2^2), (21^2), (1^4)\} \tag{4}$$

Note that the number of repetitions of a given part is often indicated by a superscript m_i where m_i is the number of parts of λ that are equal to i and will be referred to as the *multiplicity* of i in λ.

Note that in writing Eq.(4) we have given the partitions in *reverse lexicographic ordering* This ordering is such that for a pair of partitions (λ, μ) either $\lambda, \equiv \mu$ or the first non-vanishing difference $\lambda_i - \mu_i$ is *positive.*

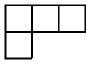
Frames of Partitions

We may associate with any partition λ a *frame* $\mathcal{F}^\lambda$ which consists of $\ell(\lambda)$, left-adjusted rows of boxes with the $i - th$ row containing λ_i boxes. Thus for $\mathcal{P}_4$ we have:-

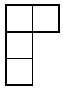


Conjugate Partitions

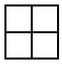
The *conjugate* of a partition λ is a partition λ' whose diagram is the transpose of the diagram of λ. If $\lambda' \equiv \lambda$ then the partition λ is said to be *self-conjugate.* Thus



and



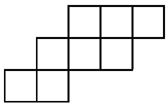
are conjugates while



is self-conjugate.

Skew Frames

Given two partitions λ and μ such that $\lambda \supset \mu$ implies that the frame F^{λ} contains the frame F^{μ}, i.e. that $\lambda_i \geq \mu_i$ for all $i \geq 1$. The difference $\rho = \lambda - \mu$ forms a *skew* frame $F^{\lambda/\mu}$. Thus, for example, the skew frame $F^{542/21}$ has the form



Note that a skew frame may consist of disconnected pieces.

Frobenius Notation for Partitions

There is an alternative notation for partitions due to Frobenius. The *diagonal* of nodes in a Ferrers-Sylvester diagram beginning at the top left-hand corner is called the *leading diagonal.* The number of nodes in the leading diagonal is called the *rank* of the partition. If r is the rank of a partition then let a_i be the number of nodes to the right of the leading diagonal in the $i-$th row and let b_i be the number of nodes below the leading diagonal in the $i-$th column. The partition is then denoted by Frobenius as

$$\begin{pmatrix} a_1, & a_2, & \ldots, & a_r \\ b_1, & b_2, & \ldots, & b_r \end{pmatrix} \tag{3.3}$$

We note that

$$a_1 > a_2 > \ldots > a_r$$
$$b_1 > b_2 > \ldots > b_r$$

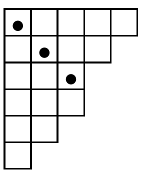
and

$$a_1 + a_2 + \ldots + a_r + b_1 + b_2 + \ldots + b_r + r = n$$

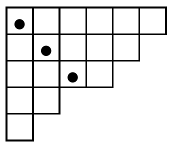
The partition conjugate to that of Eq.(3.3) is just

$$\begin{pmatrix} b_1, & b_2, & \ldots, & b_r \\ a_1, & a_2, & \ldots, & a_r \end{pmatrix} \tag{5}$$

As an example consider the partitions $(5\,4\,3^2\,2\,1)$ and $(6\,5\,4\,2\,1)$. Drawing their diagrams and marking their leading diagonal we have



and



from which we deduce the respective Frobenius designations

$$\begin{pmatrix} 4 & 2 & 0 \\ 5 & 3 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 5 & 3 & 1 \\ 4 & 2 & 0 \end{pmatrix}$$

Young Tableaux

A Young tableau is an assignment of n numbers to the n cells of a frame F^λ with $\lambda \vdash n$ according to some numbering sequence. A tableau is *standard* if the assignment of the numbers $1, 2, \ldots, n$ is such that the numbers are positively increasing from left to right in rows and down columns from top to bottom. Thus for the partitions of the integer 4 we have the standard Young tableaux

1	2	3	4

1	2	3
4		

1	2	4
3		

1	3	4
2		

1	2
3	4

1	3
2	4

1	2
3	
4	

1	3
2	
4	

1	4
2	
3	

1
2
3
4

We notice in the above examples that the number of standard tableaux for conjugate partitions is the same. Indeed the number of standard tableaux associated with a given frame F^λ is the *dimension* f_n^λ of an irreducible representation $\{\lambda\}$ of the symmetric group $\mathcal{S}_n$.

Hook lengths and dimensions for $\mathcal{S}_n$

The *hook length* of a given box in a frame F^λ is the length of the right-angled path in the frame with that box as the upper left vertex. For example, the hook length of the marked box in

*	·	·	·	
·				
·				
·				
·				

is 8.

Theorem 1: *To find the dimension of the representation of $\mathcal{S}_n$ corresponding to the frame F^λ, divide $n\,!$ by the factorial of the hook length of each box in the first column of F^λ and multiply by the difference of each pair of such hook lengths.*

Thus for the partition $(5\,4\,3^2\,2\,1)$ we have the hook lengths

10				
8				
6				
5				
3				
1				

and hence a dimension

$$\begin{aligned} f_{18}^{543^2 21} &= 18\,!\frac{2\times 4\times 5\times 7\times 9\times 2\times 3\times 5\times 7\times 1\times 3\times 5\times 2\times 4\times 2}{10\,!\times 8\,!\times 6\,!\times 5\,!\times 3\,!\times 1\,!} \\ &= 10720710 \end{aligned}$$

It is not suggested that you check the above result by explicit enumeration!

Hook-length Product $H^{\{\lambda\}}$

The irreps $\{\lambda\}$ of S_n are indexed by the ordered partitions $\lambda \vdash N$. It is useful to define a hook-length product

$$H^{\{\lambda\}} = \prod_{(i,j)\ni\lambda} h_{ij} \tag{6}$$

where i labels rows and j columns. Note that

$$H^{\{\lambda\}} = H^{\{\lambda\prime\}} \tag{7}$$

The Frame-Robinson-Thrall Formula

The $\mathcal{S}_n$ dimensional formula may be rewritten as

$$f_n^{\lambda} = \frac{n\,!}{H^{\{\lambda\}}} \tag{8}$$

which is the celebrated result of Frame, Robinson and Thrall.

Specialisation to Two-Row Irreps of $\mathcal{C}_n$

Consider a two-part partition (p, r). It is readily seen from the definition of $H^{\{\lambda\}}$ that

$$H^{\{p,r\}} = \frac{r!\,(p+1)!}{p-r+1} \tag{9}$$

Noting that $n = p + r$ we may specialise Eq. (8) to

$$f^{\{p,r\}} = \frac{p-r+1}{p+r+1}\begin{pmatrix} p+r+1 \\ r \end{pmatrix} \tag{10}$$

In quantum chemistry the Pauli exclusion principle restricts physically realisable irreps of $\mathcal{S}_n$ to the generic type $\{\frac{N}{2}+S, \frac{N}{2}-S\}$ where N and S are the total electron number and spin respectively. In that case Eq. (10) becomes

$$f^{(N,S)} = \frac{2S+1}{N+1}\begin{pmatrix} N+1 \\ \frac{N}{2}-S \end{pmatrix} \tag{11}$$

which is sometimes called the Heisenberg formula.

Staircase Partitions

A partition of the form $(p, p-1, p-2, \ldots, 2, 1)$ is termed a *staircase partition.* Such irreps have many interesting properties.

Exercises

- Show that the $p-th$ staircase partition is of weight

$$\frac{p(p+1)}{2} \tag{12}$$

- Show that the hooklength product H^p for the $p-th$ staircase partition is

$$H^p = \prod_{i=0}^{p-1}(2i+1)^{p-i} \tag{13}$$

- Show that the $p = 18$ staircase represention is of

$$353,630,151,029,664,166,403,885,519,184,771,102,250,561,450,895,264,176,910$$
$$,003,150,360,627,549,788,542,182,043,325,740,180,684,537,821,357,203,782,730$$
$$,400,746,242,708,749,607,205,510,228,035,502,080$$

- How long would it take a supercomputer to check this result by explicit computation?