Analogies between finite-dimensional irreducible representations of $\mathrm{SO}(2n)$ and infinite-dimensional irreducible representations of $\mathrm{Sp}(2n,\mathbb{R})$. I. Characters and products

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The analogy between the finite-dimensional spin representation Δ of $\mathrm{SO}(2n)$ and the infinite-dimensional representation $\tilde{\Delta}$ of $\mathrm{Sp}(2n,\mathbb{R})$ is made precise. It is then shown that this analogy can be extended so as to provide a precise link between each finite dimensional unitary irreducible representation of $\mathrm{SO}(2n)$ and a corresponding infinite-dimensional unitary irreducible representation of $\mathrm{Sp}(2n,\mathbb{R})$. The analogy shows itself at the level of the corresponding characters and difference characters, and involves the use of Schur function methods to express both characters and difference characters of $\mathrm{SO}(2n)$ and $\mathrm{Sp}(2n,\mathbb{R})$ in terms of characters of irreducible representations of their common subgroup $\mathrm{U}(n)$. The analogy is extended still further to cover the explicit decomposition of not only tensor products of Δ and $\tilde{\Delta}$ with other unitary irreducible representations of $\mathrm{SO}(2n)$ and $\mathrm{Sp}(2n,\mathbb{R})$, respectively, but also arbitrary tensor powers of Δ and $\tilde{\Delta}$. [S0022-2488(00)00307-8]

I. INTRODUCTION

The symplectic group $\mathrm{Sp}(6,\mathbb{R})$ is well known as the dynamical group of the isotropic three-dimensional harmonic oscillator.[1] For a single particle the even-parity states span a single infinite-dimensional irreducible representation commonly denoted[2,3] as $\langle \frac{1}{2}(0)\rangle$ (or $\tilde{\Delta}_+$) while the odd-parity states span the irreducible representation $\langle \frac{1}{2}(1)\rangle$ (or $\tilde{\Delta}_-$). Collectively they span a single irreducible representation $\tilde{\Delta}$ of the metaplectic group $\mathrm{Mp}(6)$, the covering group of $\mathrm{Sp}(6,\mathbb{R})$. In general the group $\mathrm{Sp}(2n,\mathbb{R})$ is of relevance to symplectic models of nuclei[4] and certain mesoscopic systems such as quantum dots.[5,6] A central problem in making applications is the resolution of tensor powers of the irreducible representation $\tilde{\Delta}$. The tensor powers of the basic irreducible representation $\tilde{\Delta}$ of $\mathrm{Sp}(2n,\mathbb{R})$ have some properties closely analogous to those of the basic spin representations of the special orthogonal group $\mathrm{SO}(2n)$. The objective of this paper is to demonstrate and exploit a variety of close analogies between irreducible representations of $\mathrm{SO}(2n)$ and $\mathrm{Sp}(2n,\mathbb{R})$.

The study of spin representations of the orthogonal groups in a space of arbitrary dimension was initiated by Brauer and Weyl. In their seminal paper[7] on this topic they showed that in the case of the orthogonal group $\mathrm{O}(2n)$ the spin representation of dimension 2^n with character Δ decomposes on restriction to the proper orthogonal group $\mathrm{SO}(2n)$ into a direct sum of two

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irreducible representations each of dimension 2^{n-1} with characters Δ_+ and Δ_-. Murnaghan[8] introduced the notion of a difference spin character Δ''. The characters of the spin representations Δ and $\Delta_\pm$, together with that of Δ'', were given by Littlewood.[9] The relevant formulas for SO($2n$) take the form:

$$\Delta=\Delta_+ + \Delta_- = \prod_{i=1}^{n} (x_i^{1/2}+x_i^{-1/2}), \tag{1.1a}$$

$$\Delta''=\Delta_+ - \Delta_- = \prod_{i=1}^{n} (x_i^{1/2}-x_i^{-1/2}), \tag{1.1b}$$

where x_i and x_i^{-1} for $i=1,2,\ldots,n$ are the eigenvalues of an arbitrary group element of SO($2n$). At the identity element I we have $x_i=1$ for $i=1,2,\ldots,n$ so that $\dim\Delta=2^n$ while $\dim\Delta''=0$.

Just as the spin representations of SO($2n$) are double-valued and are true representations of the covering group Spin($2n$), so the symplectic group Sp($2n,\mathbb{R}$) possesses certain metaplectic representations which are double-valued and are true representations of the covering group Mp($2n$). These metaplectic representations are encountered in the study of the one-dimensional harmonic oscillator and its generalizations and are variously known as harmonic representations,[3] oscillator representations,[10] and Segal–Shale–Weil representations.[11] They are the infinite-dimensional lowest weight representations associated with even and odd parity states of the harmonic oscillator. Their characters are denoted here by $\tilde{\Delta}_+$ and $\tilde{\Delta}_-$, respectively, and in what follows it will be shown that formal expressions for the sum and difference of these characters of infinite-dimensional irreducible representations of Sp($2n,\mathbb{R}$) are given by

$$\tilde{\Delta}=\tilde{\Delta}_+ + \tilde{\Delta}_- = \prod_{i=1}^{n} (x_i^{-1/2}-x_i^{1/2})^{-1}, \tag{1.2a}$$

$$\tilde{\Delta}''=\tilde{\Delta}_+ - \tilde{\Delta}_- = \prod_{i=1}^{n} (x_i^{-1/2}+x_i^{1/2})^{-1}, \tag{1.2b}$$

where now x_i and x_i^{-1} for $i=1,2,\ldots,n$ are the eigenvalues of an arbitrary group element of Sp($2n,\mathbb{R}$).

Some progress has been made on the calculation of various plethysms, that is to say symmetrized powers, of not only Δ, $\Delta_\pm$, and Δ'', but also $\tilde{\Delta}$, $\tilde{\Delta}_\pm$, and $\tilde{\Delta}''$. In particular the symmetric and antisymmetric squares of Δ and Δ'' are given by[12,13]

$$\Delta\otimes\{2\}=[1^n]_+ + [1^n]_- + \sum_{x=0}^{\infty} ([1^{n-1-4x}]+[1^{n-3-4x}]+2[1^{n-4-4x}]), \tag{1.3a}$$

$$\Delta\otimes\{1^2\}=\sum_{x=0}^{\infty} ([1^{n-1-4x}]+2[1^{n-2-4x}]+[1^{n-3-4x}]), \tag{1.3b}$$

$$\Delta''\otimes\{2\}=[1^n]_+ + \sum_{x=0}^{\infty} (-1)^{1+x}[1^{n-1-x}], \tag{1.3c}$$

$$\Delta''\otimes\{1^2\}=[1^n]_- + \sum_{x=0}^{\infty} (-1)^{1+x}[1^{n-1-x}], \tag{1.3d}$$

where $[1^k]$ are the characters of the kth fold antisymmetrized power of the defining irreducible representation $[1]$ of $\mathrm{SO}(2n)$. These representations are irreducible for $k=1,2\ldots,n-1$, while for $k=n$ we have $[1^n]=[1^n]_+ + [1^n]_-$.

Similarly, the symmetric squares of $\tilde{\Delta}$ and $\tilde{\Delta}''$ are given by[14–16]

$$\tilde{\Delta}\otimes\{2\}=\langle 1(0)\rangle+\sum_{x=0}^{\infty}\langle 1(1+x)\rangle, \tag{1.4a}$$

$$\tilde{\Delta}\otimes\{1^2\}=\langle 1(0)\rangle^*+\sum_{x=0}^{\infty}\langle 1(1+x)\rangle, \tag{1.4b}$$

$$\tilde{\Delta}''\otimes\{2\}=\langle 1(0)\rangle+\langle 1(0)\rangle^*+\sum_{x=0}^{\infty}(-\langle 1(1+4x)\rangle-\langle 1(3+4x)\rangle+2\langle 1(4+4x)\rangle), \tag{1.4c}$$

$$\tilde{\Delta}''\otimes\{1^2\}=\sum_{x=0}^{\infty}(-\langle 1(1+4x)\rangle+2\langle 1(2+4x)\rangle-\langle 1(3+4x)\rangle), \tag{1.4d}$$

where $\langle 1(m)\rangle$ are characters of certain harmonic series infinite-dimensional irreducible representations of $\mathrm{Sp}(2n,\mathbb{R})$ and an asterisk (*) signifies the associate[16] of an irreducible representation of $\mathrm{Sp}(2n,\mathbb{R})$.

Comparison of (1.1) and (1.2) gives a formal connection between the characters Δ and Δ'' of $\mathrm{SO}(2n)$ and the characters $\tilde{\Delta}$ and $\tilde{\Delta}''$ of $\mathrm{Sp}(2n,\mathbb{R})$. The formal connection is brought home rather forcibly in (1.3) and (1.4) through an analogy between the symmetrized squares of Δ and Δ'', and those of $\tilde{\Delta}$ and $\tilde{\Delta}''$. To be more precise, the analogies are between Δ and $\tilde{\Delta}''$ and between Δ'' and $\tilde{\Delta}$. Furthermore the right-hand sides of (1.3) and (1.4) signify additional analogies between $[1^n]_+$ and $\langle 1(0)\rangle$, between $[1^n]_-$ and $\langle 1(0)\rangle^*$, and, finally, between $[1^{n-t}]$ and $\langle 1(t)\rangle$ for $t>0$.

These are but the tip of an iceberg. The full set of analogies between the finite-dimensional irreducible representations of $\mathrm{SO}(2n)$ and the infinite-dimensional irreducible representations of $\mathrm{Sp}(2n,\mathbb{R})$ which we wish to expose here take the form

$$[m^n/\lambda']\leftrightarrow\langle m(\lambda)\rangle \quad \text{if} \quad \lambda_1'=m, \tag{1.5a}$$

$$[m^n/\lambda']_{+(-)^n}\leftrightarrow\langle m(\lambda)\rangle \quad \text{if} \quad \lambda_1'<m, \tag{1.5b}$$

$$[m^n/\lambda']_{-(-)^n}\leftrightarrow\langle m(\lambda)\rangle^* \quad \text{if} \quad \lambda_1'<m, \tag{1.5c}$$

$$[\Delta;m^n/\lambda']_{+(-)^n}\leftrightarrow\langle\tilde{\Delta};m(\lambda)\rangle \quad \text{if} \quad \lambda_1'\leqslant m, \tag{1.5d}$$

$$[\Delta;m^n/\lambda']_{-(-)^n}\leftrightarrow\langle\tilde{\Delta};m(\lambda)\rangle^* \quad \text{if} \quad \lambda_1'\leqslant m, \tag{1.5e}$$

where the notation used here to specify the characters of the various irreducible representations of $\mathrm{SO}(2n)$ and $\mathrm{Sp}(2n,\mathbb{R})$ will be explained fully in later sections.

The analogies (1.5) are made precise by evaluating the relevant characters of both $\mathrm{SO}(2n)$ and $\mathrm{Sp}(2n,\mathbb{R})$ at the level of their maximal compact subgroup $\mathrm{U}(n)$. The characters of irreducible representations of $\mathrm{U}(n)$ are themselves S-functions $\{\lambda\}$, and in the context of (1.5) a crucial role is played by various infinite series of S-functions.[13,17–19]

The first step in this direction is made in Sec. II by expressing each of the characters Δ, Δ'', $\tilde{\Delta}$, and $\tilde{\Delta}''$ in terms of S-function series. The second step is that of generalizing these results to the case of all the characters appearing on both sides of (1.5). As a means to this end, relevant notational devices, both algebraic and diagrammatic, are introduced in Sec. III. These are then used in Sec. IV in reformulating the known branching rules for the decomposition of irreducible representations of both SO$(2n)$ and Sp$(2n,\mathbb{R})$ on restriction to U(n). By virtue of some quite subtle S-function identities[19] and modification rules[20] precise analogies of the form (1.5) are arrived at.

It then comes as no surprise that the analogy at the level of characters between the finite-dimensional irreducible representations of SO$(2n)$ and the infinite-dimensional irreducible representations of Sp$(2n,\mathbb{R})$ can be built upon to establish analogies between the decompositions for each of these groups of tensor products, tensor powers, and symmetrized tensor powers, known as plethysms. This development is initiated in Sec. V where we content ourselves with establishing results for certain tensor products and powers involving Δ, Δ'', $\tilde{\Delta}$, and $\tilde{\Delta}''$. The extension to the case of plethysms, generalizing (1.3) and (1.4), is to be the subject of a separate paper.

II. BASIC SPIN DIFFERENCE CHARACTERS AND HARMONIC CHARACTERS

In the case of both SO$(2n)$ and Sp$(2n,\mathbb{R})$ the characters of their irreducible representations may conveniently be obtained by expressing them in terms of characters of irreducible representations of their maximal reductive subgroup U(n). The covariant tensor irreducible representations of U(n) are specified by partitions $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_n)$ into no more than n nonvanishing parts. Their characters $\{\lambda\}$, are just the Schur functions $s_\lambda(x_1,x_2,\ldots,x_n)$ of the eigenvalues x_i of the relevant group element A of U(n). The contravariant tensor irreducible representations of U(n) are just the contragredients of the covariant irreducible representations. They have characters $\{\overline{\lambda}\}$ given by $s_\lambda(x_1^{-1},x_2^{-1},\ldots,x_n^{-1})$. The particular one-dimensional irreducible representation of U(n) in which each group element A is mapped to $(\det A)^r$ for some fixed rational number r has character ϵ^r where $\epsilon=\{1^n\}=s_{1^n}(x_1,x_2,\ldots x_n)=x_1x_2\cdots x_n$.

With this notation, the characters of the two basic spin irreducible representations of SO$(2n)$ are given by

$$\Delta_+=\epsilon^{-1/2}(\{1^n\}+\{1^{n-2}\}+\{1^{n-4}\}+\cdots), \tag{2.1a}$$

$$\Delta_-=\epsilon^{-1/2}(\{1^{n-1}\}+\{1^{n-2}\}+\{1^{n-4}\}+\cdots). \tag{2.1b}$$

Similarly, the characters of the two basic harmonic irreducible representations of Sp$(2n,\mathbb{R})$ are given by

$$\tilde{\Delta}_+=\epsilon^{1/2}(\{0\}+\{2\}+\{4\}+\cdots), \tag{2.2a}$$

$$\tilde{\Delta}_-=\epsilon^{1/2}(\{1\}+\{3\}+\{5\}+\cdots). \tag{2.2b}$$

Setting $\Delta=\Delta_++\Delta_-$ and $\Delta''=\Delta_+-\Delta_-$ we have

$$\Delta=\epsilon^{-1/2}(\{0\}+\{1\}+\{1^2\}+\cdots+\{1^n\}), \tag{2.3a}$$

$$\Delta''=(-1)^n\epsilon^{-1/2}(\{0\}-\{1\}+\{1^2\}+\cdots+(-1)^n\{1^n\}). \tag{2.3b}$$

In the same way, setting $\tilde{\Delta}=\tilde{\Delta}_++\tilde{\Delta}_-$ and $\tilde{\Delta}''=\tilde{\Delta}_+-\tilde{\Delta}_-$ we have

$$\tilde{\Delta}=\epsilon^{1/2}(\{0\}+\{1\}+\{2\}+\cdots), \tag{2.4a}$$

$$\tilde{\Delta}''=\epsilon^{1/2}(\{0\}-\{1\}+\{2\}-\cdots). \tag{2.4b}$$

The use of the generating functions[9]

$$Q=\sum_{m=0}^{n}\{1^m\}=\prod_{x=1}^{n}(1+x_i), \tag{2.5a}$$

$$L=\sum_{m=0}^{n}(-1)^m\{1^m\}=\prod_{x=1}^{n}(1-x_i), \tag{2.5b}$$

$$M=\sum_{m=0}^{\infty}\{m\}=\prod_{x=1}^{n}(1-x_i)^{-1}, \tag{2.5c}$$

$$P=\sum_{m=0}^{\infty}(-1)^m\{m\}=\prod_{x=1}^{n}(1+x_i)^{-1}, \tag{2.5d}$$

in (2.3) and (2.4), together with the fact that $\epsilon^{\pm 1/2}=\Pi_{i=1}^{n}x_i^{\pm 1/2}$, then leads to the character formulas

$$\Delta=\epsilon^{-1/2}Q=\prod_{i=1}^{n}(x_i^{1/2}+x_i^{-1/2}), \tag{2.6a}$$

$$\Delta''=(-1)^n\epsilon^{-1/2}L=\prod_{i=1}^{n}(x_i^{1/2}-x_i^{-1/2}), \tag{2.6b}$$

and

$$\tilde{\Delta}=\epsilon^{1/2}M=\prod_{i=1}^{n}(x_i^{-1/2}-x_i^{1/2})^{-1}, \tag{2.7a}$$

$$\tilde{\Delta}''=\epsilon^{1/2}P=\prod_{i=1}^{n}(x_i^{-1/2}+x_i^{1/2})^{-1}, \tag{2.7b}$$

where in each case the final expressions are the ones quoted in (1.1) and (1.2). Formally, the passage from (2.4) to (1.2) as in (2.7) depends on the convergence of M and P. This requires $|x_i|<1$ for all $i=1,2,\ldots,n$. Thus (1.2a) and (1.2b) are to be viewed as formal expressions which when expanded in positive powers of x_i for all $i=1,2,\ldots,n$ define the sum and difference of the basic harmonic characters of $\mathrm{Sp}(2n,\mathbb{R})$.

It should perhaps be pointed out that Δ_+ and Δ_- are the characters of the irreducible representations of $\mathrm{SO}(2n)$ corresponding to fundamental finite-dimensional highest weight irreducible representations of the underlying simple Lie algebra D_n. Their highest weights in the fundamental weight basis, the ω-basis, and the Euclidean orthonormal basis, the ε-basis, are given by[21]

$$\Delta_+:\quad \omega_{n-1}=[\tfrac{1}{2},\tfrac{1}{2},\ldots,\tfrac{1}{2},\tfrac{1}{2}], \tag{2.8a}$$

$$\Delta_-:\quad \omega_{n}=[\tfrac{1}{2},\tfrac{1}{2},\ldots,\tfrac{1}{2},-\tfrac{1}{2}]. \tag{2.8b}$$

On the other hand, $\tilde{\Delta}_+$ and $\tilde{\Delta}_-$ are the characters of irreducible representations of $\mathrm{Sp}(2n,\mathbb{R})$ corresponding to nonfundamental lowest weight irreducible representations of the underlying simple Lie algebra C_n. Their lowest weights are given in the ω and ε bases by[10]

$$\tilde{\Delta}_+:\quad \tfrac{1}{2}\omega_n=\langle\tfrac{1}{2},\tfrac{1}{2},\ldots,\tfrac{1}{2},\tfrac{1}{2}\rangle, \tag{2.9a}$$

$$\tilde{\Delta}_-:\quad -\omega_{n-1}+\tfrac{3}{2}\omega_n=\langle\tfrac{1}{2},\tfrac{1}{2},\ldots,\tfrac{1}{2},\tfrac{3}{2}\rangle. \tag{2.9b}$$

The fact that the components of the lowest weights in the ω-basis are not integers is an indication of the fact that the corresponding irreducible representations are infinite-dimensional.

Of course there also exist highest weight irreducible representations with characters $\overline{\tilde{\Delta}}_+$ and $\overline{\tilde{\Delta}}_-$ that are contragredient to those irreducible representations having characters $\tilde{\Delta}_+$ and $\tilde{\Delta}_-$. The highest weights of these contragredient irreducible representations are given by

$$\overline{\tilde{\Delta}}_+ : \quad -\tfrac{1}{2}\omega_n = \langle -\tfrac{1}{2}, -\tfrac{1}{2}, \ldots, -\tfrac{1}{2}, -\tfrac{1}{2} \rangle, \tag{2.10a}$$

$$\overline{\tilde{\Delta}}_- : \quad \omega_{n-1} - \tfrac{3}{2}\omega_n = \langle -\tfrac{1}{2}, -\tfrac{1}{2}, \ldots, -\tfrac{1}{2}, -\tfrac{3}{2} \rangle, \tag{2.10b}$$

and their characters take the following form:

$$\overline{\tilde{\Delta}}_+ = \epsilon^{-1/2}(\{\overline{0}\} + \{\overline{2}\} + \{\overline{4}\} + \cdots), \tag{2.11a}$$

$$\overline{\tilde{\Delta}}_- = \epsilon^{-1/2}(\{\overline{1}\} + \{\overline{3}\} + \{\overline{5}\} + \cdots). \tag{2.11b}$$

As usual, setting $\overline{\tilde{\Delta}} = \overline{\tilde{\Delta}}_+ + \overline{\tilde{\Delta}}_-$ and $\overline{\tilde{\Delta}}'' = \overline{\tilde{\Delta}}_+ - \overline{\tilde{\Delta}}_-$ we then have

$$\overline{\tilde{\Delta}} = \epsilon^{-1/2}(\{\overline{0}\} + \{\overline{1}\} + \{\overline{2}\} + \cdots), \tag{2.12a}$$

$$\overline{\tilde{\Delta}}'' = \epsilon^{-1/2}(\{\overline{0}\} - \{\overline{1}\} + \{\overline{2}\} + \cdots). \tag{2.12b}$$

Replacing x_i by x_i^{-1} in the generating functions (2.5c) and (2.5d) to give $\overline{M}$ and $\overline{P}$, respectively, and using these in (2.12) then yields formulas almost identical to those of (1.2), namely

$$\overline{\tilde{\Delta}} = \epsilon^{-1/2}\overline{M} = \prod_{i=1}^{n} (x_i^{1/2} - x_i^{-1/2})^{-1}, \tag{2.13a}$$

$$\overline{\tilde{\Delta}}'' = \epsilon^{-1/2}\overline{P} = \prod_{i=1}^{n} (x_i^{1/2} + x_i^{-1/2})^{-1}. \tag{2.13b}$$

Formally, once again, the passage from (2.12) to (2.13) depends on the convergence of $\overline{M}$ and $\overline{P}$. This requires $|x_i^{-1}| < 1$ for all $i = 1,2,\ldots,n$. Thus the final formulas of (2.13a) and (2.13b) are to be viewed as formal expressions which when expanded in negative powers of x_i for all $i = 1,2,\ldots,n$ define the sum and difference of the contragredients of the basic harmonic characters of Sp($2n$,$\mathbb{R}$).

III. PARTITIONS, YOUNG DIAGRAMS, AND *S*-FUNCTIONS

Before attempting to establish the existence of analogies of the type analogies (1.5) it is necessary to develop a number of notational niceties. These are based on the use of partitions to specify a variety of Young diagrams, both standard and nonstandard, as well as corresponding S-functions and series of S-functions.[9]

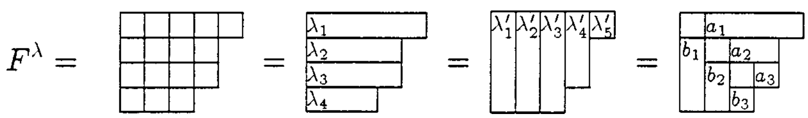
Each partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)$ of weight $|\lambda|$ specifies a Young diagram F^λ consisting of $|\lambda|$ boxes arranged in $p = l(\lambda)$ left-adjusted rows of lengths λ_i for $i = 1,2,\ldots,p$. The lengths λ_j' for $j = 1,2,\ldots,q$ of the $q = b(\lambda)$ top-adjusted columns of F^λ serve to define the conjugate partition $\lambda' = (\lambda_1', \lambda_2', \ldots, \lambda_q')$. The number of boxes $r = r(\lambda)$ on the principal diagonal of F^λ is known as the Frobenius rank of the partition λ. In Frobenius notation

$$\lambda = \begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix},$$

where a_i and b_i for $i=1,2,\ldots,r$ are the arm and leg lengths, respectively, of F^λ with respect to its main diagonal of length r. Typically, for $\lambda=(5443)=(54^23)$, with $p=\ell(\lambda)=4$ and $q=b(\lambda)=5$, we have $\lambda'=(44431)=(4^331)$ and in Frobenius notation

$$\lambda=\begin{pmatrix} 4 & 2 & 1 \\ 3 & 2 & 1 \end{pmatrix}$$

with $r=r(\lambda)=3$ and $|\lambda|=16$. This is illustrated diagrammatically by



(3.1)

As has already been seen in Sec. II partitions, λ, have a useful role to play in specifying the characters, $\{\lambda\}$, of corresponding irreducible representations of $\mathrm{U}(n)$, where these characters are S-functions. In what follows, in addition to the S-function series Q, L, M, and P, defined in (2.5), we encounter several others:[9,13,17–19]

$$A=\sum_{\alpha\in\mathcal{A}}(-1)^{|\alpha|/2}\{\alpha\},\quad B=\sum_{\beta\in\mathcal{B}}\{\beta\},\quad C=\sum_{\gamma\in\mathcal{C}}(-1)^{|\gamma|/2}\{\gamma\},\quad D=\sum_{\delta\in\mathcal{D}}\{\delta\},\tag{3.2a}$$

$$E=\sum_{\varepsilon\in\mathcal{E}}(-1)^{(|\varepsilon|+r(\varepsilon))/2}\{\varepsilon\},\quad G=\sum_{\varepsilon\in\mathcal{E}}(-1)^{(|\varepsilon|-r(\varepsilon))/2}\{\varepsilon\},\tag{3.2b}$$

$$V=\sum_{\xi\in\mathcal{X}}(-1)^{\xi'_2}\{\xi\},\quad X=\sum_{\xi\in\mathcal{X}}\{\xi\},\tag{3.2c}$$

where, in the notation exemplified in (3.1), α, β, γ, δ, ε, and ξ are characterized by the conditions $b_k=a_k+1$ for $k=1,2,\ldots,r(\alpha)$, β'_j even for $j=1,2,\ldots,b(\beta)$, $a_k=b_k+1$ for $k=1,2,\ldots,r(\gamma)$, δ_i even for $i=1,2,\ldots,\ell(\delta)$, $a_k=b_k$ for $k=1,2,\ldots,r(\varepsilon)$, and $|\xi|$ even, with $b(\xi)\leqslant 2$.

The S-function series satisfy the following conjugacy conditions:

$$A'=C,\quad B'=D,\quad E'=E,\quad G'=G,\quad M'=Q,\quad L'=P,\tag{3.3}$$

and the identities:

$$AB=CD=EG=LM=PQ=1,\quad AX=C^r,\quad AV=C,\quad AQ=G,\quad AL=E=G^r,\tag{3.4}$$

where the superscript r on any S-function series S indicates that S^r is obtained from S by mutiplying each term $\{\sigma\}$ in S by $(-1)^{r(\sigma)}$, where $r(\sigma)$ is the Frobenius rank of of σ.

Now let μ be a partition into no more than p parts with its largest part no greater than q, and let ν be any other partition. Then define $F^{\mu,_p\nu}$ to be the diagram formed by placing F^ν immediately below the pth row of F^μ, and define $F^{\mu|_q\nu}$ to be the diagram formed by placing F^ν immediately to the right of the qth column of F^μ. In the first case all the rows are left-adjusted to the same vertical line, and in the second case the columns are top-adjusted to the same horizontal line. The corresponding S-functions are denoted by $\{\mu,_p\nu\}$ and $\{\mu;_q\nu\}$. However the diagrams $F^{\mu,_p\nu}$ and $F^{\mu|_q\nu}$ may not be standard and in such cases it will be necessary to reorder their rows and columns, respectively, in accordance with the repeated use of the following modification rules:[9,22]

$$\{\lambda_1,\ldots,\lambda_i,\lambda_{i+1},\ldots\}=-\{\lambda_1,\ldots,\lambda_{i+1}-1,\lambda_i+1,\ldots\}\quad\text{for}\quad i=1,2,\ldots,\tag{3.5a}$$

$$\{\lambda'_1,\ldots,\lambda'_j,\lambda'_{j+1},\ldots\}'=-\{\lambda'_1,\ldots,\lambda'_{j+1}-1,\lambda'_j+1,\ldots\}'\quad\text{for}\quad j=1,2,\ldots\ .\tag{3.5b}$$

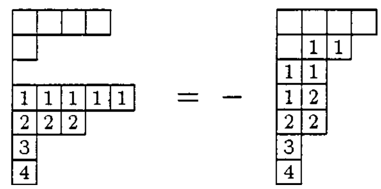
These modification rules owe their existence to the following determinantal expansions:

$$\{\lambda\}=\{\lambda'\}'=|\{\lambda_i-i+j\}|=|\{1^{\lambda'_j-j+i}\}|, \tag{3.6}$$

where as many trailing zeros as one may wish may be added to the parts of λ and λ'.

In the present context, the repeated use of (3.5a) is illustrated in the case $\mu=(41)$, $p=3$, and $\nu=(531^2)$ by

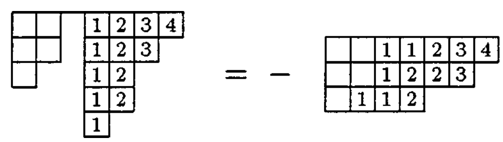
$$\{41,_3 531^2\} = -\{432^3 1^2\} \ : \tag{3.7}$$



where the labeling of boxes has been used to emphasize the fact that the modification rules may be realized by wrapping the various rows of F^ν around F^μ in the form of continuous strips. Each strip contributes a sign factor $(-1)^x$, where x is the increase in the number of rows the strip occupies as a result of the wrapping process. If the wrapping process leads to a nonstandard diagram then the result is null, that is $\{\mu,_p\nu\}=0$.

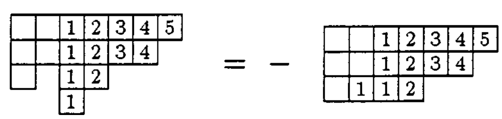
Similarly if $\mu=(2^21)$, $q=3$, and $\nu=(432^21)$ then the use of (3.5b) leads to

$$\{2^21;_3 432^21\} = -\{764\} \ : \tag{3.8}$$



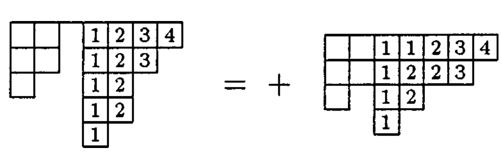
where now it is the columns of F^ν that are wrapped around F^μ in the form of continuous strips. Each strip contributes a sign factor $(-1)^y$, where y is the increase in the number of columns the strip occupies as a result of the wrapping process. As a second example of this type it is instructive to consider as before $\mu=(2^21)$ but now $q=2$ and $\nu=(5421)$:

$$\{2^21;_2 5421\} = -\{764\} \ : \tag{3.9}$$



The significance of this example is that it is possible to view the passage in (3.8) from $\{2^21;_2432^21\}$ to $-\{764\}$ as one from $\{2^21;_2432^21\}$ first to $\{2^21;_25421\}$ and then, as in (3.9), from $\{2^21;_25421\}$ to $-\{764\}$. The first step just involves sliding the portion of F^ν below the main diagonal one step in a northwesterly direction:

$$\{2^21;_3 432^21\} = \{2^21;_2 5421\} \ : \tag{3.10}$$



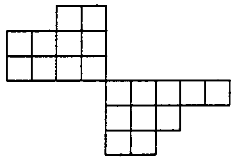
The identity $\{2^21;_3432^21\}=\{2^21;_25421\}$ illustrated in (3.10) involves the partition (432^21) $\in\mathcal{A}$, and its conjugate $(5421)\in\mathcal{C}$, where the sets $\mathcal{A}$ and $\mathcal{C}$ are those associated with the S-function series A and C, respectively, defined in (3.2a). The result (3.10) can be generalized immediately to the case of all $\alpha\in\mathcal{A}$, or equivalently all $\gamma\in\mathcal{C}$, giving the identity

$$\{\mu;_q\gamma\}=(-1)^{r(\alpha)}\{\mu;_{q+1}\alpha\} \quad \text{with} \quad \alpha=\gamma'\in\mathcal{A}, \tag{3.11}$$

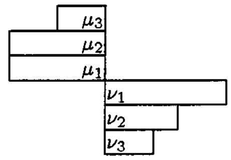
which will be used in Sec. IV.

It is also necessary to recall that each pair of partitions μ and ν specifies a composite Young diagram $F^{\overline{\mu};\nu}$ which may be variously drawn as shown:

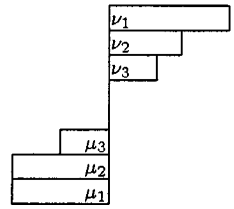
$F^{\overline{\mu};\nu}$ =



=



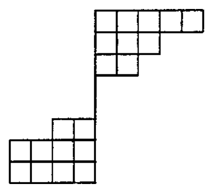
or



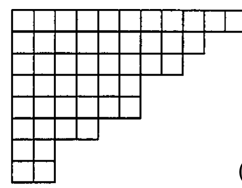
(3.12)

where the last form involves a total of precisely n rows within the context, as here, of characters of U(n). In fact for any $q\geqslant\mu_1$, the corresponding character $\{\overline{\mu};\nu\}$ of U(n) is given by $\epsilon^{-q}\{\lambda\}$, where F^{λ} is formed from $F^{\overline{\mu};\nu}$ by taking the complement of the portion F^{μ} in an $n\times q$ rectangle and placing it the left of F^{ν} to give $F^{(q^n/\mu);_q\nu}$. Typically, in U(8) and choosing $q=6$, we have

$\{\overline{4^22};532\} = \epsilon^{-6}\{11,986^242^2\}$:



$= \epsilon^{-6}$



(3.13)

More generally, our notation for characters of U(n) is such that for any $q\geqslant\mu_1$ we have

$$\{\overline{\mu};\nu\}=\epsilon^{-q}\{(q^n/\mu);_q\nu\}, \tag{3.14}$$

where it may be necessary to invoke a modification of the type illustrated in (3.8) in order to standardize the final result.

IV. BRANCHING RULES FOR SO(2*n*)→U(*n*) AND SP(2*N*,ℝ)→U(*N*)

The branching rules for the restrictions from SO$(2n)$ to U(n) and from Sp$(2n,\mathbb{R})$ to U(n) appear at first sight to have little in common. For example, in the case of the restriction SO$(2n)\to$U(n) it is known that if λ is a partition into fewer than n parts, then[17]

$$[\lambda]\to\sum_{\zeta} \{\overline{\zeta};\lambda/B\zeta\}, \tag{4.1}$$

where the summation is over all partitions ζ for which λ/ζ is nonzero, where the slash (/) signifies a quotient of Schur functions, and quite generally $\{\overline{\mu};\nu\}$ signifies the character of an irreducible mixed tensor irreducible representation of U(n). On the other hand, for the restriction from Sp$(2n,\mathbb{R})\to$U(n) we have[3]

$$\langle \tfrac{1}{2}k(\lambda)\rangle\to\epsilon^{k/2}\cdot\{\lambda_s\}^k\cdot D, \tag{4.2}$$

where $\{\lambda_s\}^k$ is a signed sequence of Schur functions $\pm\{\mu\}$ such that $[\mu]$ is equivalent to $\pm[\lambda]$ under the modification rules of O(k). Here each Schur function $\{\mu\}$ is the character of an irreducible covariant tensor irreducible representation of U(n).

The complete set of inequivalent unitary finite-dimensional irreducible representations of SO$(2n)$ have characters which may conveniently be specified by

$$[\lambda] \quad \text{for} \quad \lambda_1' < n, \quad [\lambda]_\pm \quad \text{for} \quad \lambda_1' = n, \quad [\Delta;\lambda]_\pm \quad \text{for} \quad \lambda_1' \leqslant n. \tag{4.3}$$

The relevant branching rules for the restriction from SO(2n) to U(n) which serve to define these characters take the form[18]

$$[\lambda] = \sum_\xi \{\overline{\xi};\lambda/\xi B\} \quad \text{for} \quad \lambda_1' < n, \tag{4.4a}$$

$$[\Box;\mu] = \sum_\xi \epsilon^{-1}\{\overline{\xi};(\mu/\xi B)\cdot X\} \quad \text{for} \quad \mu_1' \leqslant n, \tag{4.4b}$$

$$[\Box;\mu]'' = (-1)^n \sum_\xi \epsilon^{-1}\{\overline{\xi};(\mu/\xi B)\cdot V\} \quad \text{for} \quad \mu_1' \leqslant n, \tag{4.4c}$$

$$[\Delta;\lambda] = \sum_\xi \epsilon^{-1/2}\{\overline{\xi};(\lambda/\xi B)\cdot Q\} \quad \text{for} \quad \lambda_1' \leqslant n, \tag{4.4d}$$

$$[\Delta;\lambda]'' = (-1)^n \sum_\xi \epsilon^{-1/2}\{\overline{\xi};(\lambda/\xi B)\cdot L\} \quad \text{for} \quad \lambda_1' \leqslant n, \tag{4.4e}$$

where

$$[\lambda]_\pm = [\Box;\mu]_\pm = \tfrac{1}{2}([\Box;\mu] \pm [\Box;\mu]'') \quad \text{for} \quad \lambda_1' = n, \tag{4.5a}$$

$$[\Delta;\lambda]_\pm = \tfrac{1}{2}([\Delta;\lambda] \pm [\Delta;\lambda]'') \quad \text{for} \quad \lambda_1' \leqslant n, \tag{4.5b}$$

and in the case $\lambda_1' = n$ it has been convenient to write $[\lambda] = [1^n;{}_1\mu] = [\Box;\mu]$ with $\mu_1' \leqslant n$.

In order to rewrite the formulas (4.4) in a form more suited to the exposure of the analogies we are seeking it is necessary to invoke the following:

Lemma 4.1: Let λ *be an arbitrary partition and* S *an arbitrary* S*-function series, then with* B *as in* (3.2a),

$$\sum_\xi \{\overline{\xi};(\lambda/\xi B)S\} = \{\overline{\lambda};AS\}\cdot B, \tag{4.6}$$

where $A = B^{-1}$.

Proof: The crucial observation is that, as shown elsewhere,[19] for all partitions ζ we have $D/\zeta = (\zeta/D)D$. Either the use of an entirely analogous argument with D replaced by B, or by the simpler expedient of taking conjugates, one deduces that $B/\zeta = (\zeta/B)B$. Using this together with the fact that $BA = 1$ allows us to derive (4.6) as follows:

$$\sum_\xi \{\overline{\xi};(\lambda/\xi B)S\} = \sum_\zeta \{\overline{\lambda/\zeta};(\zeta/B)BAS\} = \sum_\zeta \{\overline{\lambda/\zeta};(B/\zeta)AS\} = \sum_\zeta \{\overline{\lambda};AS\}\cdot B, \tag{4.7}$$

where the last step depends on the linear extension from σ to B of the product rule

$$\{\overline{\mu};\nu\}\cdot\{\sigma\} = \sum_\zeta \{\overline{\mu/\zeta};\nu\cdot(\sigma/\zeta)\}. \tag{4.8}$$

Applying Lemma 4.1 to (4.4), and using the identities (3.4), gives

$$[\lambda] = \{\overline{\lambda};A\}\cdot B \quad \text{for} \quad \lambda_1' < n, \tag{4.9a}$$

$$[\Box;\mu]=\epsilon^{-1}\{\overline{\mu};C^r\}\cdot B \quad \text{for} \quad \mu_1'\leqslant n, \tag{4.9b}$$

$$[\Box;\mu]''=(-1)^n\epsilon^{-1}\{\overline{\mu};C\}\cdot B \quad \text{for} \quad \mu_1'\leqslant n, \tag{4.9c}$$

$$[\Delta;\lambda]=\epsilon^{-1/2}\{\overline{\lambda};G\}\cdot B \quad \text{for} \quad \lambda_1'\leqslant n, \tag{4.9d}$$

$$[\Delta;\lambda]''=(-1)^n\epsilon^{-1/2}\{\overline{\lambda};G^r\}\cdot B \quad \text{for} \quad \lambda_1'\leqslant n. \tag{4.9e}$$

Introducing m with $m\geqslant\lambda_1$, we can then use the notation of (3.14) to arrive at the formulas:

$$[\lambda]=\epsilon^{-m}\{(m^n/\lambda);_mA\}\cdot B \quad \text{for} \quad \lambda_1'<n, \tag{4.10a}$$

$$[\lambda]=\epsilon^{-m}\{(m^n/\lambda);_mA\}\cdot B \quad \text{for} \quad \lambda_1'=n, \tag{4.10b}$$

$$[\lambda]''=(-1)^n\epsilon^{-m}\{(m^n/\lambda);_mA^r\}\cdot B \quad \text{for} \quad \lambda_1'=n, \tag{4.10c}$$

$$[\Delta;\lambda]=\epsilon^{-m-1/2}\{(m^n/\lambda);_mG\}\cdot B \quad \text{for} \quad \lambda_1'\leqslant n, \tag{4.10d}$$

$$[\Delta;\lambda]''=(-1)^n\epsilon^{-m-1/2}\{(m^n/\lambda);_mG^r\}\cdot B \quad \text{for} \quad \lambda_1'\leqslant n. \tag{4.10e}$$

In the case of the passage from (4.9b) and (4.9c) to (4.10b) and (4.10c), respectively, it has also been necessary to note that, as a consequence of (3.11), for all $\gamma\in\mathcal{C}$ we have

$$\begin{aligned}\{\overline{\mu};\gamma\}&=\epsilon^{-m+1}\ \{((m-1)^n/\mu);_{m-1}\gamma\}=(-1)^{r(\alpha)}\epsilon^{-m+1}\{((m-1)^n/\mu);_m\alpha\}\\&=(-1)^{r(\alpha)}\epsilon^{-m+1}\ \{(m^n/(1^n\cdot\mu);_m\alpha\}=(-1)^{r(\alpha)}\epsilon^{-m+1}\ \{(m^n/\lambda;_m\alpha\},\end{aligned} \tag{4.11}$$

with $\alpha\in\mathcal{A}$.

It follows from (4.10) and (4.5) that

$$[\lambda]=\epsilon^{-m}\{(m^n/\lambda);_mA\}\cdot B \quad \text{for} \quad \lambda_1'<n, \tag{4.12a}$$

$$[\lambda]_+=\epsilon^{-m}\{(m^n/\lambda);_mA^{eo(n)}\}\cdot B \quad \text{for} \quad \lambda_1'=n, \tag{4.12b}$$

$$[\lambda]_-=\epsilon^{-m}\{(m^n/\lambda);_mA^{oe(n)}\}\cdot B \quad \text{for} \quad \lambda_1'=n, \tag{4.12c}$$

$$[\Delta;\lambda]_+=\epsilon^{-m-1/2}\{(m^n/\lambda);_mG^{eo(n)}\}\cdot B \quad \text{for} \quad \lambda_1'\leqslant n, \tag{4.12d}$$

$$[\Delta;\lambda]_-=\epsilon^{-m-1/2}\{(m^n/\lambda);_mG^{oe(n)}\}\cdot B \quad \text{for} \quad \lambda_1'\leqslant n, \tag{4.12e}$$

or equivalently

$$[m^n/\lambda']=\epsilon^{-m}\{\lambda';_mA\}\cdot B \quad \text{for} \quad \lambda_1'=m, \tag{4.13a}$$

$$[m^n/\lambda']_+=\epsilon^{-m}\{\lambda';_mA^{eo(n)}\}\cdot B \quad \text{for} \quad \lambda_1'<m, \tag{4.13b}$$

$$[m^n/\lambda']_-=\epsilon^{-m}\{\lambda';_mA^{oe(n)}\}\cdot B \quad \text{for} \quad \lambda_1'<m, \tag{4.13c}$$

$$[\Delta;m^n/\lambda']_+=\epsilon^{-m-1/2}\{\lambda';_mG^{eo(n)}\}\cdot B \quad \text{for} \quad \lambda_1'\leqslant m, \tag{4.13d}$$

$$[\Delta;m^n/\lambda']_-=\epsilon^{-m-1/2}\{\lambda';_mG^{oe(n)}\}\cdot B \quad \text{for} \quad \lambda_1'\leqslant m, \tag{4.13e}$$

where λ has been replaced by m^n/λ' and use has been made of the fact that $(m^n/(m^n/\lambda'))=\lambda'$. For all Schur function series S we have introduced $S^{eo(n)}$ and $S^{oe(n)}$ such that

$$S^{eo(n)}=\begin{cases} S^e & \text{if } n \text{ is even} \\ S^o & \text{if } n \text{ is odd;} \end{cases}$$

and

$$S^{oe(n)}=\begin{cases} S^o & \text{if } n \text{ is even} \\ S^e & \text{if } n \text{ is odd,} \end{cases} \tag{4.14}$$

where S^e and S^o are the terms of even and odd Frobenius rank, respectively, in the Schur function series S.

The remarkable analogy beween these character formulas for SO$(2n)$ and those of Sp$(2n,\mathbb{R})$ is exposed by recalling that the signed sequences $\{\lambda_s\}^k$ in (4.2) can be expressed rather succinctly in terms of our series of Schur functions. This has its origin in Newell's formulation[20,23] of the modification rules of O(k). One has to distinguish both between even and odd values of k, and for even k between those partitions λ having fewer than $k/2$ or precisely $k/2$ parts. The relevant expressions have been given by Rowe *et al.*[2] When used in (4.2) they imply

$$\langle m(\lambda)\rangle=\epsilon^m \ \{\lambda,{}_mC\}\cdot D \quad \text{for} \quad \lambda_1'=m, \tag{4.15a}$$

$$\langle m(\lambda)\rangle=\epsilon^m \ \{\lambda,{}_mC^e\}\cdot D \quad \text{for} \quad \lambda_1'<m, \tag{4.15b}$$

$$\langle m(\lambda)\rangle^*=\epsilon^m \ \{\lambda,{}_mC^o\}\cdot D \quad \text{for} \quad \lambda_1'<m, \tag{4.15c}$$

$$\langle\tilde{\Delta};m(\lambda)\rangle=\epsilon^{m+1/2} \ \{\lambda,{}_mG^e\}\cdot D \quad \text{for} \quad \lambda_1'\leqslant m, \tag{4.15d}$$

$$\langle\tilde{\Delta};m(\lambda)\rangle^*=\epsilon^{m+1/2} \ \{\lambda,{}_mG^o\}\cdot D \quad \text{for} \quad \lambda_1'\leqslant m, \tag{4.15e}$$

where as before C^e and C^o are the even and odd Frobenius rank terms in C, while G^e and G^o are the even and odd Frobenius rank terms in G. In (4.15d) and (4.15e) it has also been convenient in the case $k=2m+1$ to denote $\langle k/2(\lambda)\rangle$ by $\langle\tilde{\Delta};m(\lambda)\rangle$ in order to emphasize the analogies with (4.13d) and (4.13e).

The analogy (1.5) that we were seeking between the finite-dimensional irreducible representations of SO$(2n)$ and the infinite-dimensional irreducible representations of Sp$(2n,\mathbb{R})$ has thus been made explicit through the analogous expressions (4.13) and (4.15) for the corresponding characters. To summarize, the analogy involves replacing the partition λ by the complement of its conjugate with respect to (m^n) on the left-hand sides, replacing ϵ^p by ϵ^{-p} and taking conjugates on the right-hand sides, noting the conjugacy relations $A'=C$, $B'=D$ and $(\mu;{}_m\nu)'=(\mu',{}_m\nu')$, and taking care to distinguish between the cases for which n is even and odd.

V. TENSOR PRODUCTS

An earlier study[13] of the decomposition of tensor or Kronecker products of irreducible representations of SO$(2n)$ has revealed that

$$\Delta\times[\lambda]=[\Delta;\lambda/Q]_+ + [\Delta;\lambda/Q]_- \quad \text{for} \quad \lambda_1'<n, \tag{5.1a}$$

$$\Delta\times[\lambda]_+=[\Delta;\lambda/Q]_+ \quad \text{for} \quad \lambda_1'=n, \tag{5.1b}$$

$$\Delta\times[\lambda]_-=[\Delta;\lambda/Q]_- \quad \text{for} \quad \lambda_1'=n, \tag{5.1c}$$

$$\Delta\times[\Delta;\lambda]_+=[(\Box;\lambda)/Q]_{(+)} \quad \text{for} \quad \lambda_1'\leqslant n, \tag{5.1d}$$

$$\Delta\times[\Delta;\lambda]_-=[(\Box;\lambda)/Q]_{(-)} \quad \text{for} \quad \lambda_1'\leqslant n, \tag{5.1e}$$

$$\Delta''\times[\lambda]=[\Delta;\lambda/L]_{+}-[\Delta;\lambda/L]_{-}\quad\text{for}\quad\lambda_1'<n, \tag{5.1f}$$

$$\Delta''\times[\lambda]_{+}=[\Delta;\lambda/L]_{+}\quad\text{for}\quad\lambda_1'=n, \tag{5.1g}$$

$$\Delta''\times[\lambda]_{-}=-[\Delta;\lambda/L]_{-}\quad\text{for}\quad\lambda_1'=n, \tag{5.1h}$$

$$\Delta''\times[\Delta;\lambda]_{+}=[(\Box;\lambda)/L]_{(+)}\quad\text{for}\quad\lambda_1'\leqslant n, \tag{5.1i}$$

$$\Delta''\times[\Delta;\lambda]_{-}=-[(\Box;\lambda)/L]_{(-)}\quad\text{for}\quad\lambda_1'\leqslant n, \tag{5.1j}$$

where

$$[\mu]_{(\pm)}=\begin{cases}[\mu] & \text{if } \mu_1'<n\\ [\mu]_{\pm} & \text{if } \mu_1'=n.\end{cases} \tag{5.2}$$

In the case of $\mathrm{Sp}(2n,\mathbb{R})$ the analogous formulas take the form

$$\tilde{\Delta}\times\langle m(\lambda)\rangle=\langle\tilde{\Delta};m(\lambda\cdot M)_m\rangle+\langle\tilde{\Delta};m(\lambda\cdot M)_m\rangle^*\quad\text{for}\quad\lambda_1'=m, \tag{5.3a}$$

$$\tilde{\Delta}\times\langle m(\lambda)\rangle=\langle\tilde{\Delta};m(\lambda\cdot M)_m\rangle\quad\text{for}\quad\lambda_1'<m, \tag{5.3b}$$

$$\tilde{\Delta}\times\langle m(\lambda)\rangle^*=\langle\tilde{\Delta};m(\lambda\cdot M)_m\rangle^*\quad\text{for}\quad\lambda_1'<m, \tag{5.3c}$$

$$\tilde{\Delta}\times\langle\tilde{\Delta};m(\lambda)\rangle=\langle m+1(\lambda\cdot M)_{m+1}\rangle\quad\text{for}\quad\lambda_1'\leqslant m, \tag{5.3d}$$

$$\tilde{\Delta}\times\langle\tilde{\Delta};m(\lambda)\rangle^*=\langle m+1(\lambda\cdot M)_{m+1}\rangle^{(*)}\quad\text{for}\quad\lambda_1'\leqslant m, \tag{5.3e}$$

$$\tilde{\Delta}''\times\langle m(\lambda)\rangle=\langle\tilde{\Delta};m(\lambda\cdot P)_m\rangle-\langle\tilde{\Delta};m(\lambda\cdot P)_m\rangle^*\quad\text{for}\quad\lambda_1'=m, \tag{5.3f}$$

$$\tilde{\Delta}''\times\langle m(\lambda)\rangle=\langle\tilde{\Delta};m(\lambda\cdot P)_m\rangle\quad\text{for}\quad\lambda_1'<m, \tag{5.3g}$$

$$\tilde{\Delta}''\times\langle m(\lambda)\rangle^*=-\langle\tilde{\Delta};m(\lambda\cdot P)_m\rangle^*\quad\text{for}\quad\lambda_1'<m, \tag{5.3h}$$

$$\tilde{\Delta}''\times\langle\tilde{\Delta};m(\lambda)\rangle=\langle m+1(\lambda\cdot P)_{m+1}\rangle\quad\text{for}\quad\lambda_1'\leqslant m, \tag{5.3i}$$

$$\tilde{\Delta}''\times\langle\tilde{\Delta};m(\lambda)\rangle^*=-\langle m+1(\lambda\cdot P)_{m+1}\rangle^{(*)}\quad\text{for}\quad\lambda_1'\leqslant m, \tag{5.3j}$$

where $(\lambda\cdot S)_p$ signifies that the product $\lambda\cdot S$ is to be evaluated in $\mathrm{U}(p)$ so that quite generally any term $(\mu)_p=0$ if $\mu_1'>p$. In addition,

$$\langle p(\mu)\rangle^{(*)}=\begin{cases}\langle p(\mu)\rangle^* & \text{if } \mu_1'<p\\ \langle p(\mu)\rangle & \text{if } \mu_1'=p.\end{cases} \tag{5.4}$$

In the case of (5.1) all the characters on the left-hand side are well defined and standard provided that $n\geqslant\lambda_1'$. Moreover, on the right every expression involves merely a quotient with Q or L leading to a finite number of terms, all of which are necessarily standard in $\mathrm{SO}(2n)$. The same cannot be said of (5.2). First of all, even if $n\geqslant\lambda_1'$ the associate characters signified by $\langle m(\lambda)\rangle^*$ and $\langle\tilde{\Delta};m(\lambda)\rangle^*$ may not be standard. In fact they will be null if $n<k-\lambda_1'$ where $k=2m$ or $2m+1$, as appropriate. Moreover, all the expressions on the right involve an infinite number of terms by virtue of their dependence on products with M or P. In addition the associate

characters on the right may be null for certain values of n for which the left-hand side is well defined. This is the case, in particular, for $n=m=\lambda_1'$ for which all terms $\langle\cdots\rangle^*$ on the right-hand side of (5.3a) and (5.3f) are null, and may be omitted.

In order to derive the results (5.3) it should be recalled[3] that

$$\langle \tfrac{1}{2}k(\lambda)\rangle\times\langle\tfrac{1}{2}j(\mu)\rangle=\sum_{\nu} R_{\nu}^{\lambda\mu}\langle\tfrac{1}{2}(k+j)(\nu)\rangle, \tag{5.5}$$

where the coefficients $R_{\nu}^{\lambda\mu}$ are the branching rule coefficients for the restriction from O$(k+j)$ to O$(k)\times$O(j):

$$\mathrm{O}(k+j)\rightarrow\mathrm{O}(k)\times\mathrm{O}(j):\quad [\nu]\rightarrow\sum_{\lambda\mu} R_{\nu}^{\lambda\mu}[\lambda]\times[\mu]. \tag{5.6}$$

In the special case of interest here we require $j=1$ with μ equal to (0) or (1) for which $[\mu]$ is $[0]$ or $[1]=[0]^*$, respectively. The corresponding branching rule takes the form:

$$\mathrm{O}(k+1)\rightarrow\mathrm{O}(k)\times\mathrm{O}(1):\quad [\nu]\rightarrow\sum_{m=0}^{\infty}[\nu/m]\times[0]^{(*)^m}, \tag{5.7}$$

where $[0]^{(*)^m}=[0]$ or $[0]^*$ according to whether m is even or odd, respectively. It then follows from (5.5) and (5.6) that

$$\langle\tfrac{1}{2}k(\lambda)\rangle\times\langle\tfrac{1}{2}(0)\rangle=\sum_{m:\text{even}}\langle\tfrac{1}{2}(k+1)(\lambda\cdot m)\rangle, \tag{5.8a}$$

$$\langle\tfrac{1}{2}k(\lambda)\rangle\times\langle\tfrac{1}{2}(0)\rangle^*=\sum_{m:\text{odd}}\langle\tfrac{1}{2}(k+1)(\lambda\cdot m)\rangle. \tag{5.8b}$$

Recalling that $\tilde{\Delta}=\langle\tfrac{1}{2}(0)\rangle+\langle\tfrac{1}{2}(0)\rangle^*$ and $\tilde{\Delta}''=\langle\tfrac{1}{2}(0)\rangle-\langle\tfrac{1}{2}(0)\rangle^*$, and taking care over the lengths of the various partitions appearing in $\lambda\cdot M$ and $\lambda\cdot P$ and the distinction between a character and its associate, one arrives at (5.3a)–(5.3j) for $k=2m$ and $2m+1$ as appropriate.

If further evidence is needed of the close parallel between finite-dimensional irreducible representations of SO(2n) and infinite-dimensional irreducible representations of Sp(2n,$\mathbb{R}$) it is provided by the following rather striking branching rule formulas.

First, it has been shown by Morris[24,25] that Δ and Δ'' decompose as follows under the appropriate restriction:

$$\mathrm{SO}(4mn)\rightarrow\mathrm{SO}(2n)\times\mathrm{O}(2m):$$
$$\Delta\rightarrow\sum_{\lambda:l(\lambda)<m}([m^n/\lambda']_+\times[\lambda]+[m^n/\lambda']_-\times[\lambda]^*)+\sum_{\lambda:l(\lambda)=m}[m^n/\lambda']\times[\lambda], \tag{5.9a}$$

$$\mathrm{SO}(4mn)\rightarrow\mathrm{SO}(2n)\times\mathrm{O}(2m):$$
$$\Delta''\rightarrow\sum_{\lambda:l(\lambda)<m}(-1)^{|\lambda|}([m^n/\lambda']_+\times[\lambda]+[m^n/\lambda']_-\times[\lambda]^*)$$
$$+\sum_{\lambda:l(\lambda)=m}(-1)^{|\lambda|}[m^n/\lambda']\times[\lambda], \tag{5.9b}$$

$\mathrm{SO}(4mn+2n)\rightarrow\mathrm{SO}(2n)\times\mathrm{O}(2m+1)$:

$$\Delta\rightarrow\sum_{\lambda:l\leqslant m}([\Delta;m^n/\lambda']_+\times[\lambda]+[\Delta;m^n/\lambda']_-\times[\lambda]^*), \tag{5.9c}$$

$\mathrm{SO}(4mn+2n)\rightarrow\mathrm{SO}(2n)\times\mathrm{O}(2m+1)$:

$$\Delta''\rightarrow\sum_{\lambda:l(\lambda)\leqslant m}(-1)^{|\lambda|}([\Delta;m^n/\lambda']_+\times[\lambda]-[\Delta;m^n/\lambda']_-\times[\lambda]^*). \tag{5.9d}$$

Second, the defining property of $\tilde{\Delta}$ which encapsulates the fact that $\mathrm{Sp}(2n,\mathbb{R})$ and $\mathrm{O}(k)$ are a complementary pair of mutually centralizing subgroups of $\mathrm{Sp}(2nk,\mathbb{R})$ takes the form[3]

$$\mathrm{Sp}(2nk,\mathbb{R})\rightarrow\mathrm{Sp}(2n,\mathbb{R})\times O(k):\quad \tilde{\Delta}\rightarrow\sum_{\lambda}\langle\tfrac{1}{2}k(\lambda)\rangle\times[\lambda]. \tag{5.10}$$

Setting $k=2m$ and $2m+1$ in turn, and consideration of $\tilde{\Delta}''$, then yields, in direct analogy to (5.1), the following results:

$\mathrm{Sp}(4mn,\mathbb{R})\rightarrow\mathrm{Sp}(2n,\mathbb{R})\times\mathrm{O}(2m)$:

$$\begin{aligned}\tilde{\Delta}&\rightarrow\sum_{\lambda}\langle m(\lambda)\rangle\times[\lambda]\\ &=\sum_{\lambda:l(\lambda)<m}(\langle m(\lambda)\rangle\times[\lambda]+\langle m(\lambda)\rangle^*\times[\lambda]^*)+\sum_{\lambda:l(\lambda)=m}\langle m(\lambda)\rangle\times[\lambda],\end{aligned} \tag{5.11a}$$

$\mathrm{Sp}(4mn,\mathbb{R})\rightarrow\mathrm{Sp}(2n,\mathbb{R})\times\mathrm{O}(2m)$:

$$\begin{aligned}\overline{\Delta''}&\rightarrow\sum_{\lambda}(-1)^{|\lambda|}\langle m(\lambda)\rangle\times[\lambda]\\ &=\sum_{\lambda:l(\lambda)<m}(-1)^{|\lambda|}(\langle m(\lambda)\rangle\times[\lambda]+\langle m(\lambda)\rangle^*\times[\lambda]^*)\\ &\quad+\sum_{\lambda:l(\lambda)=m}(-1)^{|\lambda|}\langle m(\lambda)\rangle\times[\lambda],\end{aligned} \tag{5.11b}$$

$\mathrm{Sp}(4mn+2n,\mathbb{R})\rightarrow\mathrm{Sp}(2n,\mathbb{R})\times\mathrm{O}(2m+1)$:

$$\begin{aligned}\overline{\Delta}&\rightarrow\sum_{\lambda}\langle\tilde{\Delta};m(\lambda)\rangle\times[\lambda]\\ &=\sum_{\lambda:l(\lambda)\leqslant m}(\langle\overline{\Delta;}m(\lambda)\rangle\times[\lambda]+\langle\tilde{\Delta};m(\lambda)\rangle^*\times[\lambda]^*),\end{aligned} \tag{5.11c}$$

$\mathrm{Sp}(4mn+2n,\mathbb{R})\rightarrow\mathrm{Sp}(2n,\mathbb{R})\times\mathrm{O}(2m+1)$:

$$\begin{aligned}\overline{\Delta''}&\rightarrow\sum_{\lambda}(-1)^{|\lambda|}\langle\tilde{\Delta};m(\lambda)\rangle\times[\lambda]\\ &=\sum_{\lambda:l(\lambda)\leqslant m}(-1)^{|\lambda|}(\langle\tilde{\Delta};m(\lambda)\rangle\times[\lambda]-\langle\overline{\Delta;}m(\lambda)\rangle^*\times[\lambda]^*).\end{aligned} \tag{5.11d}$$

These results offer us the opportunity of decomposing arbitrary, kth-fold, tensor powers of Δ, Δ'', $\tilde{\Delta}$, and $\tilde{\Delta}''$. This is exemplified in the case of Δ and $k=2m$ through a consideration of the group–subgroup chains:

$$\mathrm{SO}(4mn)\begin{matrix}\nearrow \mathrm{SO}(2n)\times\mathrm{SO}(2n)\times\cdots\times\mathrm{SO}(2n)\searrow\\ \searrow \mathrm{SO}(2n)\times\mathrm{O}(2m)\nearrow\end{matrix}\mathrm{SO}(2n) \tag{5.12}$$

for which we have the branching rules

$$\Delta\begin{matrix}\nearrow \Delta\times\Delta\times\cdots\times\Delta\searrow\\ \searrow \sum_\lambda [m^n/\lambda']\times[\lambda]\nearrow\end{matrix}(\Delta)^{\times 2m}=\sum_\lambda \dim_{2m}[\lambda][m^n/\lambda']. \tag{5.13}$$

To derive the identity on the right-hand side one merely proceeds from the SO($4mn$) character Δ to its SO($2n$) content by both upper and lower routes. From the definition (1.1a) of Δ one can introduce a set of $2mn$ parameters x_{ia} for $i=1,2,\ldots,n$ and $a=1,2,\ldots,2m$ to give

$$\Delta=\prod_{i=1}^{n}\prod_{a=1}^{2m}(x_{ia}^{1/2}+x_{ia}^{-1/2}). \tag{5.14}$$

The upper route involves setting $x_{ia}=x_i$ for all i and a to give

$$\Delta\to\prod_{a=1}^{2m}\left(\prod_{i=1}^{n}(x_{ia}^{1/2}+x_{ia}^{-1/2})\right)\to\left(\prod_{i=1}^{n}(x_i^{1/2}+x_i^{-1/2})\right)^{2m}=(\Delta)^{2m}, \tag{5.15}$$

while the lower route depends first on the use of the branching rule (5.9a) in which context it is convenient to set $x_{ia}=x_i y_a$ for all i and a, where x_i and y_a denote eigenvalues of groups elements of SO($2n$) and O($2m$), respectively, and then allowing all y_a to take the value 1. This corresponds to restricting O($2m$) to its identity element. Using this in (5.9a) then gives the factor $\dim_{2m}[\lambda]$ appearing in (5.13).

Proceeding in an exactly similar way, but this time from the definition (1.2a) of $\tilde{\Delta}$, we obtain by consideration of the group–subgroup chains

$$\mathrm{Sp}(2nk)\begin{matrix}\nearrow \mathrm{Sp}(2n)\times\mathrm{Sp}(2n)\times\cdots\times\mathrm{Sp}(2n)\searrow\\ \searrow \mathrm{Sp}(2n)\times\mathrm{O}(2m)\nearrow\end{matrix}\mathrm{Sp}(2n), \tag{5.16}$$

the branching rule identity

$$\tilde{\Delta}\begin{matrix}\nearrow \tilde{\Delta}\times\tilde{\Delta}\times\cdots\times\tilde{\Delta}\searrow\\ \searrow \sum_\lambda \langle m(\lambda)\rangle\times[\lambda]\nearrow\end{matrix}(\tilde{\Delta})^{\times 2m}=\sum_\lambda \dim_{2m}[\lambda]\langle m(\lambda)\rangle. \tag{5.17}$$

Generalizing to the case $k=2m+1$, and extending these results to both Δ'' and $\tilde{\Delta}''$ we obtain the following complete set of formulas for the decomposition of tensor powers of Δ, Δ'', $\tilde{\Delta}$, and $\tilde{\Delta}''$:

$$(\Delta)^{\times 2m}=\sum_\lambda \dim_{2m}[\lambda][m^n/\lambda'], \tag{5.18a}$$

$$(\Delta'')^{\times 2m}=\sum_\lambda (-1)^{|\lambda|}\dim_{2m}[\lambda][m^n/\lambda']', \tag{5.18b}$$

$$(\Delta)^{\times(2m+1)}=\sum_{\lambda} \dim_{2m+1}[\lambda][\Delta;m^n/\lambda'], \tag{5.18c}$$

$$(\Delta'')^{\times(2m+1)}=\sum_{\lambda} (-1)^{|\lambda|}\dim_{2m+1}[\lambda][\Delta;m^n/\lambda'], \tag{5.18d}$$

and

$$(\tilde{\Delta})^{\times 2m}=\sum_{\lambda} \dim_{2m}[\lambda]\langle m(\lambda)\rangle, \tag{5.19a}$$

$$(\tilde{\Delta}'')^{\times 2m}=\sum_{\lambda} (-1)^{|\lambda|}\dim_{2m}[\lambda]\langle m(\lambda)\rangle'', \tag{5.19b}$$

$$(\tilde{\Delta})^{\times(2m+1)}=\sum_{\lambda} \dim_{2m+1}[\lambda]\langle\tilde{\Delta};m(\lambda)\rangle, \tag{5.19c}$$

$$(\tilde{\Delta}'')^{\times(2m+1)}=\sum_{\lambda} (-1)^{|\lambda|}\dim_{2m+1}[\lambda]\langle\tilde{\Delta};m(\lambda)\rangle''. \tag{5.19d}$$

Of these results, (5.18a) and (5.18c) were first given by Bauer,[26] and (5.19a) and (5.19b) were given by Kashiwara and Vergne,[11] but the others are new. The results themselves and their mode of derivation all serve to confirm the depth and significance of the analogies spelled out in (1.5).

One can go still further and decompose our kth-fold powers into their various symmetrized powers known as plethysms whose symmetry is specified by partitions κ of k. This task is deferred to part II of the present work.

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