Exceptional Lie groups in physics

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The role of the exceptional Lie groups in physics is reviewed. Methods of calculating properties of exceptional Lie groups are considered and illustrated by a number of examples.

1. INTRODUCTION

The exceptional Lie groups first appeared, just over 100 years ago, in the *Theses* of Elie Cartan[1]. Cartan successfully obtained a complete classification of the complex semisimple Lie groups and their associated Lie algebras*.

Cartan found four series of Lie algebras which he designated as A_ℓ, B_ℓ, C_ℓ and D_ℓ which he associated with the classical Lie groups $SU_{\ell+1}$, $SO_{2\ell+1}$, $Sp_{2\ell}$ and $SO_{2\ell}$. The letter ℓ was used to designate the rank of the Lie algebra which was the dimension of the maximal Abelian subalgebra. Cartan cast the commutation relations for semisimple Lie algebras into the standard form

$$[H_i, H_j] = 0 \quad (i, k = 1, \ldots, \ell) \tag{1.a}$$

$$[H_i, E_\alpha] = \alpha_i E_\alpha \tag{1.b}$$

$$[E_\alpha, E_\beta] = N_{\alpha\beta} E_{\alpha+\beta} \quad (\text{if } \alpha + \beta \neq 0) \tag{1.c}$$

$$[E_\alpha, E_{-\alpha}] = \alpha^i H_i \tag{1.d}$$

* It is interesting to note that even before Cartan's classification two English lawyers and two Russians independently established what is today known as the E_8 root lattice. Such lattices are relevant to various aspects of coding theory. For a popular account see N. A. Sloane, Sci. Amer. **250**,(11), 116 (1984)

There is a Lie algebra for each of the four series of classical Lie algebras for each value of the positive integer ℓ. In addition Cartan found five Lie algebras that occurred for only five special ranks 2,4,6,7 and 8. These he termed exceptional Lie algebras and designated them by the letter labels G_2, F_4, E_6, E_7 and E_8.

It is interesting to note that even before

The exceptional groups appeared to be of little, if any, interest to physicists for some 55 years after Cartan's thesis when Guilio Racah[2] introduced physicists to the exceptional group G_2 in his analysis of the $f-$shell where he recognised the existence of G_2 as a subgroup of SO_7 and exploited it in both the classification of states and giving an elegant account of the Coulomb and spin-orbit interactions. In doing this Racah was mindful of the importance of his analysis for the interpretation of the very complex spectra of the rare earths. Indeed he titled his series of four papers "The Theory of Complex Spectra". His work initially made little impact on the atomic physics community who were by and large satisfied with the book of Condon and Shortley[3] which required no knowledge of the theory of Lie groups. The subject of Racah algebra was taken up intensely by nuclear physicists such as Jahn[4] (of the Jahn-Teller effect) and Flowers[5], the latter using the group G_2 in his analysis of the nuclear $f-$shell. The development in the late 1950's of the experimental study of the electronic properties of rare earths in crystalline environments by optical spectroscopy and electron spin resonance techniques finally stimulated the application of Racah's theory of complex spectra to its original purpose. The seminal paper of Elliott, Judd and Runciman[6] followed by Judd's book[7] brought to a much wider group of atomic physicists the basic ideas of Racah's theory and its use of the exceptional group G_2. The same year, 1963, as the publication of Judd's book saw the publication of complete computer produced tables of matrix elements for calculations of properties of the electronic $f-$shell.

The group G_2 was used by Racah[2] simply as a mathematical device to simplify otherwise complex calculations – no physical significance was attached to the use of G_2. Nevertheless, one remarkable result arose from Racah's analysis of the Coulomb interaction for states of maximum multiplicity, independently of the values of the Slater radial integrals, their relative energies depended on a single parameter E_3 and furthermore the ${}^4S$ and ${}^4F$ terms in f^3 were degenerate as were also the term pair ${}^5S,{}^5F$ in f^4. Racah also realised that his treatment was much simpler than might reasonably been expected, commenting that "The values of $c(WW'(200))$ are given in Table XII but the results are much simpler than could be expected from that table."

There was a considerable revival of interest in the possibilities of the exceptional groups in the construction of grand-unified theories of the fundamental interaction forces as typified by the review of Gell-Mann, Ramond and Slansky[8] in 1978. A storm of interest in the exceptional group E_8 was created in 1984 by Green and Schwarz[9] dramatic development of superstring theories. The role of the exceptional groups E_6 and G_2 in describing extensions of the interacting boson model (IBM) of nuclei to $sdgi$ bosons has been considered[10] much in the style of Racah's original analysis for the $f-$shell.

Along side the development of applications of the exceptional groups has been the need to develop methods for calculating the properties of representations of the exceptional groups and their relevant subgroups and it is to this area we devote the remainder of this article.

2. LABELLING REPRESENTATIONS

The unitary representations of the compact Lie groups may be uniquely labelled by giving there maximal weight vector. The irreducible representations of the classical Lie groups SU_n, SO_n and Sp_{2n} may be equivalently labelled by partitions of integers. Littlewood[11] has give such a systematic labelling encasing the relevant partitions in $\{,\}$ brackets for SU_n, $[,]$ for $SO-n$ and $\langle,\rangle$ for Sp_{2n} with all zeroes omitted except for the case of the trivial representation. Wybourne and Bowick[12] made a systematic study of the basic properties of the exceptional Lie groups and suggested that their irreducible representations could be uniquely labelled in terms of the partitions used to label the irreducible representations of one of their maximal subgroups and in particular to the partition label of the subgroup corresponding to the highest weight irreducible representation contained in the relevant group-subgroup decomposition. The relevant group-subgroup pairs they considered for labelling purposes were $G_2 \supset SU_3$, $F_4 \supset SO_9$, $E_6 \supset SU_2 \times SU_6$, $E_7 \supset SU_8$ and $E_8 \supset SU_9$. Thus every irreducible representation was uniquely labelled by a constrained partition enclosed in $(,)$. The whole question of providing natural labels for the irreducible representations of the exceptional groups was clearly spelt out by King and Al-Qubanchi[13,14] who, while largely cohering with the labelling advocated by Wybourne and Bowick, chose in the case of the exceptional groups E_6, E_7 and E_8 to take the partition label as that of the irreducible representation of the appropriate maximal subgroup contragredient to the highest weight irreducible representation contained in the group-subgroup decomposition, a practice this author recommends. Thus whereas Wybourne and Bowick labelled the fundamental representation of E_7 as (1^6) King and Al-Qubanchi used the label (1^2). It is important to note that in the case of the group G_2 Racah's (u_1, u_2) labels are related to the corresponding SU_3 based labels (λ_1, λ_2) by

$$\lambda_1 = u_1 + u_2 \qquad \lambda_2 = u_2 \tag{2}$$

In this article we use the SU_3 based labels. King and Al-Qubanchi give detailed tables of the relationship of the natural labels to those of Dynkin [15,16] and of the modification rules required to convert non-standard partition labels into standard labels.

3. KRONECKER PRODUCTS AND BRANCHING RULES

The choice of an appropriate labelling scheme for the irreducible representations of the exceptional groups can lead to simplifications in the computation of Kronecker products and branching rules. Much effort has been put into the development of results for the classical Lie groups in terms of the properties of Schur functions or $S-$functions for brevity, (for a general review see Littlewood[11] or Wybourne[17]. Modern mathematical presentations have been given by Macdonald[18] and by Sagan[19]). These methods make extensive use of the Littlewood-Richardson rule for resolving products of $S-$functions and making use of

Young tableaux. King[21] emphasised the importance of the role of certain infinite series of $S-$functions which allowed a succint description of many branching rules involving the classical groups. King *et al*[22] went on to apply those methods to the problem of resolving symmetrised powers of rotation groups which was later to prove useful in handling the corresponding problem of resolving symmetrised powers of exceptional group representations. Central to the computation of Kronecker products for exceptional group irreducible representations was King's[23] observation that if one knew the decomposition of a given irreducible representation to those of the maximal subgroup H defining the labelling of the irreducible representations then the Kronecker product of *any* other irreducible representation with the given irreducible representation could be found by computing for the group H the Kronecker product of the decomposed irreducible representation s with that of H corresponding to the labelling of the other exceptional group irreducible representation followed by simple application of modification rules for the subgroup H followed by those for the exceptional group. Black *et al*[23] went on to give general S-function methods for handling the Kronecker products Of all the classical Lie groups and to systematise their evaluation for the exceptional groups.

The development of the above techniques allowed their implementation into the computer programme **SCHUR**[24,25] which has permited the rapid evaluation of many properties of both the classical and exceptional Lie groups as well as some information on the properties of the infinite dimensional unitary representations of the non-compact groups $Mp(n)$ and $Sp(2n, R)$ which are relevant to many-particle harmonic oscillator problems such as arise in the theory of quantum dots[26]. A very early version of **SCHUR** constructed by P. H. Butler, was used to construct extensive tables[17] that have received wide application to problems relating to the $f-$shell. The modern version has vastly greater possibilities.

4. VANISHING MATRIX ELEMENTS

As a young Ph.D. student in the late 1950's I was endeavouring to apply Racah's methods to the interpretation of spectra of rare earths in crystals and was involved in extensive hand calculations of spin-orbit matrix elements each often involving two or three pages of calculation only to find the matrix element vanished even though it satisfied all the common angular momentum selection rules. Alister McLellan, then HOD of Physics at Canterbury remarked "There must be a group-theoretical explanation for the vanishing of the spin-orbit matrix elements - selection rules tell you what you will not get not what you will get". McLellan[27] noted that the spin-orbit interaction transformed under Racah's $SO_7 \supset G_2 \supset SO_3$ group scheme used to classify the $f-$shell states as $[1^2](11)$ and constructed two tables, the first gave the number, $c(WW'[1^2])$, of times the irreducible representation $[1^2]$ of $SO(7)$ occurred in the Kronecker products $W \times W'$ where W, W' were pairs of irreducible representations of SO_7. The second table gave the corresponding numbers $c(U, U', (21))$ for the group G_2. One had then simply to check these tables and if the number was null then the matrix elements of the spin-orbit interaction involving states described by the relevant labels associated with $SO_7 \supset G_2$ were necessarily null. This immediately explained many of the observed null matrix elements, but not all.

Over many years, Judd has attempted to expose the hidden simplicities of the $f-$shell that Racah hinted at in his 1949 paper[2]. Of particular interest has been the exposing of unsuspected symmetries behind the vanishing of various matrix elements that appear to satisfy the usual selection rules and of suprising proportionalities between sets of matrix elements. The initial studies[28] involved extensions of the method introduced by McLellan observing, for example[29], that if an operator $\mathcal{O}$ transformed with respect to some group-subgroup $\mathcal{G} \supset \mathcal{H}$ according to the irrep pair $W_G U_H$ then the reduced matrix element $\langle WU || W_G U_H || W_G U_H \rangle$ would be necessarily null if W appeared in the symmetric part of the Kronecker square of the irrep W_G while U appeared in the antisymmetric part of the Kronecker square of the irrep U_H, or vice versa. This became known as the explanation by conflicting symmetries. It was this observation that led us[30,31] to look at the general problem of resolving Kronecker powers of irreducible representations into their symmetry components using Littlewood's[11] method of plethysm[17]. Even with the introduction of the method of conflicting symmetries many null matrix elements defied explanation. This was particularly apparent when effective three-body operators were introduced[32,33] and their matrix elements calculated for the $f-$shell[34].

5. JUDD'S QUARK MODEL OF THE $f-$SHELL AND SO_8

It was with that background that Judd and his associates[35−44] developed and entirely new model of the $f-$shell which they have named "the quark model" by allusion to the well known quark model of the hadrons[45]. Basically, Judd's model stems from his observation that the states of the electronic $f-$shell can be regarded as arising from the combinations of eight "quarks" that span the basic spinor irreducible representation Δ of SO_7. A pair of "quarks" are coupled together to form states of either *odd (ungerade)* or *even (gerade)* parity and belong to a particular representation W of SO_7 occurring in

$$\Delta \times \Delta = [0] + [1] + [1^2] + [1^3] \tag{3}$$

Thus the spin-up ket states may be written as

$$|(\Delta\Delta)_p W \uparrow \tau L M_L\rangle \tag{4}$$

where p is the parity label. Such kets have a well defined particle number N and spin S. Thus these kets give a complete description of states of maximum multiplicity. A general state can be formed by combining kets associated with separate spin-up and spin-down spaces, namely

$$|(\Delta\Delta)_p W_\uparrow, (\Delta\Delta)_p W_\downarrow; W \tau L M_L\rangle \tag{5}$$

Crucial to the "quark" model and much of its subsequent development is the observation that the spinor irreducible representation Δ of SO_7 can be embedded irreducibly into the vector irreducible representation [1] of the group SO_8. Furthermore, the three-fold automorphism[11,46,47] of the irreducible representations of SO_8 makes possible three distinct group structures for the description of the states of the $f-$shell. The various mappings between these different structures can lead to considerable insights into the properties of matrix operators of interactions within the $f-$shell. The automorphisms of $SO-8$ play an important role in establishing the relevant branching rules and plethysms for resolving symmetrised powers of SO_8 irreducible representations[48]. The irreducible representations of SO_8 can labelled by ordered partition labels $[a,b,c,d]$ where a,b,c,d are either all integers or all half-integers. If

$$a = b + c \qquad \text{and} \qquad d = 0 \tag{6}$$

the irreducible representation is said to have *null triality* meaning that it is unchanged by the automorphism. All other irreducible representations of SO_8 have triality. The automorphism is such that[11,22,49]

$$\begin{aligned}
&[a,b,c,d] \\
&\quad \to \left[\frac{(a+b+c+d)}{2}, \frac{(a+b-c-d)}{2}, \frac{(a-b+c-d)}{2}, \frac{(a+b+c+d)}{2}\right] \\
&\quad \to \left[\frac{(a+b+c-d)}{2}, \frac{(a+b-c+d)}{2}, \frac{(a-b+c+d)}{2}, \frac{(a-b-c-d)}{2}\right] \\
&\quad \to [a,b,c,d]
\end{aligned} \tag{7}$$

The automorphisms of SO_8 irreducible representations can often greatly simplify calculations such as symmetrised powers. For example, consider the evaluation of the SO_8 plethysm $[\Delta;1^2]_- \otimes \{21\}$ which at first sight appears formidable. But carrying out the automorphism twice we see $[\Delta;1^2]_- \to [21]$. It is not difficult to evaluate $[21] \otimes \{21\}$ in the group O_8 to obtain

$$\begin{array}{lllllll}
[21] \otimes \{21\} = & [621] & +\ [61] & +\ [54] & +\ 2[531] & +\ [52^2] & +\ 2[521^2] \\
& +\ 4[52] & +\ [51^2]\# & +\ 4[51^2] & +\ 2[5] & +\ [4^21] & +\ 3[432] \\
& +\ 3[431^2] & +\ 5[43] & +\ 3[42^21] & +\ 2[421]\# & +\ 12[421] & +\ 6[41^3] \\
& +\ 10[41] & +\ 3[3^221] & +\ [3^21]\# & +\ 8[3^21] & +\ [32^3] & +\ 2[32^2]\# \\
& +\ 8[32^2] & +\ 12[321^2] & +\ [32]\# & +\ 14[32] & +\ 4[31^2]\# & +\ 16[31^2] \\
& +\ 6[3] & +\ 5[2^31] & +\ 4[2^21]\# & +\ 14[2^21] & +\ 10[21^3] & +\ [21]\# \\
& +\ 15[21] & +\ 2[1^3]\# & +\ 6[1^3] & +\ 4[1] & &
\end{array}$$

where the hash sign, #, is used to distinguish conjugate pairs of O_8 irreducible representations. Now standardise the resultant irreducible representations as for SO_8 to yield

$$\begin{aligned}[21]\otimes\{21\} = {} & [621] + [61] + [54] + 2[531] + [52^2] + 2[521^2]_- \\ & + 2[521^2]_+ + 4[52] + 5[51^2] + 2[5] + [4^21] + 3[432] \\ & + 3[431^2]_- + 3[431^2]_+ + 5[43] + 3[42^21]_- + 3[42^21]_+ + 14[421] \\ & + 6[41^3]_- + 6[41^3]_+ + 10[41] + 3[3^221]_- + 3[3^221]_+ + 9[3^21] \\ & + [32^3]_- + [32^3]_+ + 10[32^2] + 12[321^2]_- + 12[321^2]_+ + 15[32] \\ & + 20[31^2] + 6[3] + 5[2^31]_- + 5[2^31]_+ + 18[2^21] + 10[21^3]_- \\ & + 10[21^3]_+ + 16[21] + 8[1^3] + 4[1]\end{aligned}$$

Applying the automorphism to the above result finally yields

$$\begin{aligned}[\Delta;1^2]_+\otimes\{21\} = {} & [\Delta;4^2]_+ + [\Delta;4321]_+ + 2[\Delta;431]_+ + [\Delta;43]_- + [\Delta;42^2]_+ \\ & + 2[\Delta;421^2]_+ + 3[\Delta;421]_- + 3[\Delta;42]_+ + 3[\Delta;41^2]_+ + 3[\Delta;41]_- \\ & + [\Delta;4]_+ + [\Delta;3^22^2]_+ + 2[\Delta;3^22]_+ + 4[\Delta;3^21^2]_+ + 3[\Delta;3^21]_- \\ & + 5[\Delta;3^2]_+ + 5[\Delta;32^21]_+ + 3[\Delta;32^2]_- + 3[\Delta;321^2]_- + 14[\Delta;321]_+ \\ & + 9[\Delta;32]_- + 6[\Delta;31^3]_+ + 10[\Delta;31^2]_- + 12[\Delta;31]_+ + 5[\Delta;3]_- \\ & + 2[\Delta;2^4]_+ + [\Delta;2^31]_- + 6[\Delta;2^3]_+ + 10[\Delta;2^21^2]_+ + 12[\Delta;2^21]_- \\ & + 15[\Delta;2^2]_+ + 5[\Delta;21^3]_- + 20[\Delta;21^2]_+ + 18[\Delta;21]_- + 10[\Delta;2]_+ \\ & + 6[\Delta;1^4]_+ + 10[\Delta;1^3]_- + 16[\Delta;1^2]_+ + 8[\Delta;1]_- + 4[\Delta;0]_+\end{aligned}$$

Such an exercise becomes trivial using **SCHUR** but does illustrate one of the many simplifications that arise in exploiting the automorphisms of SO_8.

6. RETURN OF THE EXCEPTIONAL GROUPS

Racah's successful exploitation of the exceptional group G_2 might lead one to conclude that that is the end of the appearance of the exceptional groups in atomic physics and there is no role for the still more exotic exceptional groups, such as F_4 and E_6, in atomic physics. Such a conclusion is perhaps too hasty. We have already noted the use of E_6 in the interacting boson model of nuclei[10]. In that case there was a natural embedding of the relevant angular momentum states s, d, g, i into the fundamental 27−dimensional irreducible representation $(1:1)$ of E_6. Furthermore, the 27−dimensional irreducible representation (2) of G_2 can be irreducibly embedded in the fundamental irreducible representation of E_6 and thus there are no spurious states.

In 1969 Wadzinski[50] considered the group F_4 in the classification of the states of an $N-$electron configuration $(s+d+g+h)^N$. While a mathematically interesting structure it is largely irrelevant to atomic physics though perhaps not outside of the province of the interacting boson model of nuclei. Judd[51,52] has considered the applicability of the Lie group F_4 to the atomic $f-$shell by associating his s and f quarks with pseudo-spins of $I=2$ and 1 respectively to permit the embedding of $SO_3^I \times G_2$ in F_4. In this way he has been able to shed further light on the unusual structure of the $f-$shell as reflected in the various relationships that are found to exist for atomic operators acting among $f-$shell states. Two group chains are of particular relevance for the orbital states L of the $f-$shell

$$F_4 \supset SO_8 \supset SO_7 \supset [G_2 \supset SO_3^L] \tag{7}$$

and

$$F_4 \supset SO_3^I \times [G_2 \supset SO_3^L] \tag{8}$$

where the $[G_2 \supset SO_3^L]$ part of the two chains are identical. A given irreducible representation of F_4 may be decomposed following either chain. While the beginning and end of the two chains are identical the intermediate portions of the two chains will, in general, be quite distinct. Thus a matrix element that satisfies the selection rules for one chain may not satisfy those arising in the other chain. Judd has further noted that since $E_6 \supset SU_3 \times G_2$ and $E_6 \supset F_4$ possibilities involving E_6 can be considered. Much work remains to be done but it is clear that efficient means of handling the properties of the exceptional groups and their subgroups are an essential prerequisite to detailed investigations. To that end one needs to be able to evaluate branching rules, Kronecker products and resolve symmetrised powers of irreducible representations for all the relevant groups and subgroups.

Relatively few general results are known for the exceptional groups. However, we give below some results we have recently established:-

$$E_6 \to U_1 \times SO_{10}$$

$$(n:0) \to \sum_{(a,b,c)} \{2a-b-4c\} \times [\frac{2a+b}{2}, \frac{b}{2}, \frac{b}{2}, \frac{b}{2}, \frac{b}{2}] \quad (a+b+c=n) \tag{9}$$

$$(2n:0) \to \{3(a-d)\} \times [\frac{a+2b+d}{2}, \frac{a+2b+d}{2}, \frac{a+d}{2}, \frac{a+d}{2}, \frac{a-d}{2}] \quad (a+b+c+d=n) \tag{10}$$

$$E_6 \to F_4$$

$$\begin{aligned}(n:n) \to& (n)+(n-1)+\ldots+(0)\\ &= (n/M)\end{aligned} \tag{11}$$

$$\begin{aligned}(2n:0) \to& (n,n)+(n,n-1)+\ldots+(n)\\ &= (n,n/M)\end{aligned} \tag{12}$$

$$F_4 \to SO_9$$

$$(n) \to \sum_{(a,b,c)} [\frac{2a+b}{2}, \frac{b}{2}, \frac{b}{2}, \frac{b}{2}] \quad (a+b+c=n) \tag{13}$$

$$(n,n) \to \sum_{(a,b)} [\frac{2a+b}{2}, \frac{2a+b}{2}, \frac{b}{2}, \frac{b}{2}] \quad (a+b=2n) \tag{14}$$

$$SO_7 \to G_2$$

$$[n] \to (n) \tag{15}$$

$$[nn] \to \sum_{m=0}^{n} (2n/m, n/m) \tag{16}$$

$$[nnn] \to (2n/M) \tag{17}$$

$$G_2 \to SU_3$$

$$(n) \to \sum_{m=0}^{n} \sum_{k=0}^{m} \{m,k\} \tag{18}$$

$$(2n,n) \to \sum_{m=0}^{n} \sum_{k=0}^{m} \{2n-m, n-k\} \tag{19}$$

The programme **SCHUR** has been used to generate much relevant information for specific representations of the exceptional groups and their subgroups. This information is too voluminous to include here but may be obtained as a TEX file from the author by e-mail. As an example, we give in Table 1 a shortened table of the decompositions of some irreducible representations of F_4 under $F_4 \to SO_3 \times G_2$. If one knows the decompositions for one group chain it is often a comparatively simple task to obtain the decompositions for an alternative chain involving the same initial and final groups. Thus suppose one knows the decompositions for the chain

$$E_8 \supset SO_16 \supset SO_9 \times SO7 \supset SO_9 \times G_2 \tag{20}$$

and wishes to obtain the decompositions for $E_8 \supset F_4 \times G_2$. These may be found by comparison of

$$E_8 \supset F4 \times G_2 \supset SO_9 \times G_2 \tag{21}$$

with the decompositions obtained from Eq (20). Thus the decomposition of the 248–dimensional irreducible representation (21^7) under Eq (20) to $SO_9 \times G_2$ yields

$$(21^7) \to ([11] + [s;0]) \times (0) + [[1] + [s;0] + [0]] \times (1) + [0] \times (21) \tag{22}$$

But under $F_4 \to SO_9$ we have

$$\begin{aligned}(11) \to& [11] + [s;0] \\ (1) \to& [1] + [s;0] + [0] \\ (0) \to& [0]\end{aligned}$$

leading immediately to the $E_8 \to F_4 \times G_2$ branching rule

$$(21^7) \to (11) \times (0) + (1) \times (1) + (0) \times (21)$$

Proceeding in that way we readily establish the decompositions shown in Table 2 which go beyond those currently in the literature.

8. BACK TO THE FUTURE

It is now possible to calculate many of the properties of the exceptional groups and their subgroups with relative ease. As a consequence it becomes possible to explore new avenues of applications of exceptional groups in physics. At the time of writing it is by no means clear the directions such studies will take. It is intriguing to see the group SO_8 emerging as a significant group in the $f-$shell and to see hints of still wider structures involving the exceptional groups with SO_8 and G_2 as significant subgroups. Perhaps the largest exceptional group, E_8, will yet make an appearance in atomic physics, along with its maximal subgroup SO_{16} which in turn contains naturally the subgroup $SO_8 \times SO_8$ or alternatively with E_8's doubly exceptional subgroup $F_4 \times G_2$. The exceptional groups, along with SU_3 may all be given constructions in terms of octonions. Perhaps these are part of the never-ending story of atomic structure.

9. A PERSONAL NOTE

As this volume is dedicated to the memory of Professor Adolfas Jucys I would like to conclude on a personal note. I became aware of the work of Jucys and his collaborators in the early 1960's and decided to travel to Vilnius to meet him in 1968. Travel was not easy to arrange but I finally reached Vilnius by train through Warsaw and there was Jucys, with car to meet us, but the Intourist representative had also arrived and insisted we travel to the hotel in the Intourist car with Jucys in pursuit from the rear. It was a memorable visit with great hospitality from Professor Jucys and his charming wife. It was great to overcome the barriers of separation that existed in those times. During that time I also met Vladas Vanagas who was well ahead of his time in his mathematical comprehension of atomic and nuclear structures, alas also no longer with us. There were the keen young students including Rudzikas, Savukynas, Glembockis and Alisauskas. Jucys saw clearly that computers would play an increasingly more significant part in future developments and I do not believe he would be surprised by these developments.

ACKNOWLEDGEMENT

The work reported herein has been supported by Polish KBN Grant 18/p3/94/07

Table 1. Branching Rules for $F_4 \to SO_3 \times G_2$

The representations of SO_3 are enclosed in square brackets and those of G_2 in curved brackets. The labels $(\lambda_1\lambda_2)$ for G_2 are based on the maximal SU_3 subgroup. The corresponding Racah labels (u_1u_2)may be found by the relationship

$$u_1 = \lambda_1 - \lambda_2, \quad u_2 = \lambda_2$$

F_4	$SO_3 \times G_2$
(1)	$[2](0)+[1](1)$
(1^2)	$[2](1)+[1](0)+[0](21)$
$(s;1)$	$[3](1)+[3](0)+[2](21)+[2](1)+[1](2)+[1](1)+[1](0)+[0](1)$
(2)	$[4](0)+[3](1)+[2](2)+[2](1)+[2](0)+[1](21)+[1](1)+[0](2)+[0](0)$
(21)	$[4](1)+[3](21)+[3](2)+[3](1)+[3](0)+[2](21)+[2](2)+2[2](1)+[2](0)$
	$+[1](31)+[1](21)+[1](2)+2[1](1)+[1](0)+[0](1)$
(21^2)	$[4](21)+[4](1)+[3](2)+[3](1)+[3](0)+[2](31)+[2](21)+[2](2)$
	$+2[2](1)+[1](21)+[1](2)+[1](1)+[1](0)+[0](3)+[0](21)+[0](1)$
(22)	$[4](2)+[3](21)+[3](1)+[2](31)+[2](2)+[2](1)+[2](0)+[1](21)+[1](1)$
	$+[0](42)+[0](2)+[0](0)$
$(s;2)$	$[5](1)+[5](0)+[4](21)+[4](2)+2[4](1)+[4](0)+[3](31)+2[3](21)+2[3](2)$
	$+3[3](1)+[3](0)+[2](31)+[2](3)+2[2](21)+3[2](2)+4[2](1)+2[2](0)+[1](31)$
	$+[1](3)+2[1](21)+3[1](2)+3[1](1)+[1](0)+[0](31)+[0](21)+[0](2)+[0](1)$

Table 2. Some $E_8 \to F_4 \times G_2$ Decompositions

$(21) \to$	$(2) \times (0)$	$+\ (s;1) \times (1)$	$+\ (1^2) \times (21)$	$+\ (1) \times (2)$	$+\ (1) \times (1)$
	$+\ (0) \times (2)$	$+\ (0) \times (0)$			
$(21^7) \to$	$(1^2) \times (0)$	$+\ (1) \times (1)$	$+\ (0) \times (21)$		
$(3) \to$	$(s;2) \times (1)$	$+\ (s;2) \times (0)$	$+\ (21^2) \times (21)$	$+\ (21) \times (2)$	$+\ (21) \times (1)$
	$+\ (2) \times (21)$	$+\ (2) \times (2)$	$+\ (2) \times (1)$	$+\ (2) \times (0)$	$+\ (s;1) \times (31)$
	$+\ (s;1) \times (21)$	$+\ (s;1) \times (2)$	$+\ 2(s;1) \times (1)$	$+\ (1^2) \times (3)$	$+\ (1^2) \times (21)$
	$+\ (1^2) \times (2)$	$+\ (1^2) \times (1)$	$+\ (1^2) \times (0)$	$+\ (1) \times (31)$	$+\ (1) \times (3)$
	$+\ (1) \times (21)$	$+\ 2(1) \times (2)$	$+\ 2(1) \times (1)$	$+\ (1) \times (0)$	$+\ (0) \times (31)$
	$+\ (0) \times (21)$	$+\ (0) \times (2)$			
$(31^6) \to$	$(21^2) \times (0)$	$+\ (21) \times (1)$	$+\ (2) \times (21)$	$+\ (2) \times (1)$	$+\ (s;1) \times (2)$
	$+\ (s;1) \times (1)$	$+\ (s;1) \times (0)$	$+\ (1^2) \times (21)$	$+\ (1^2) \times (2)$	$+\ (1^2) \times (0)$
	$+\ (1) \times (31)$	$+\ (1) \times (21)$	$+\ (1) \times (2)$	$+\ 2(1) \times (1)$	$+\ (0) \times (3)$
	$+\ (0) \times (21)$	$+\ (0) \times (1)$			
$(42^7) \to$	$(2^2) \times (0)$	$+\ (21) \times (1)$	$+\ (2) \times (2)$	$+\ (2) \times (0)$	$+\ (s;1) \times (21)$
	$+\ (s;1) \times (1)$	$+\ (1^2) \times (21)$	$+\ (1^2) \times (1)$	$+\ (1) \times (31)$	$+\ (1) \times (2)$
	$+\ (1) \times (1)$	$+\ (1) \times (0)$	$+\ (0) \times (42)$	$+\ (0) \times (2)$	$+\ (0) \times (0)$

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